

## Equal volumes of regular polytopes with commensurable edges

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### ABSTRACT

An equilateral triangle and a square can be rescaled to have the same area, but their side lengths cannot both be rational. We prove the corresponding statement for every pair of regular convex polytopes in a fixed dimension: equal volume and a rational ratio of edge lengths force congruence. For polygons, the proof uses a fractional linear expression for the cotangent that generates the full cyclotomic field. In higher dimensions, a prime valuation argument treats the simplex, cube and cross-polytope together, leaving only the exceptional three- and four-dimensional types. We also give a direct proof of a known rational-angle classification, apply it to algebraic areas of regular star polygons, and record an arithmetic criterion for equal numerical volumes in different dimensions.

**Keywords and Phrases:** Regular polytopes, polygon area, cyclotomic fields, rational angles, commensurable lengths

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## 1. Introduction

Without a restriction on scale, any two regular polygons can be made to have the same area. The question changes when their side lengths must be positive integers in a common unit. Can an equilateral triangle and a square have equal area then? Can any two different regular polygons? No, and the same is true for every dimension.

The natural hypothesis is slightly broader than integrality. Two positive lengths are *commensurable* if their ratio is rational. Equivalently, there is a choice of length unit in which both are positive integers. This formulation makes the question independent of the unit chosen.

**Theorem 1.1.** *Let  $d \geq 2$ . Two regular convex  $d$ -polytopes with commensurable edge lengths have equal  $d$ -dimensional volume if and only if they are congruent.*

For polygons, the area formula reduces the problem to a rational ratio of cotangents. The key observation is that a simple invertible change of variable turns a cotangent into a primitive root of unity. In higher dimensions, the classical classification of regular polytopes leaves three infinite families and a few exceptional types [6]. A single choice of an odd prime handles the three families together. Rational trigonometric values have a long history; see Olmsted [9] and Niven [8]. Conway and Jones [5] developed a wider theory of trigonometric Diophantine equations through vanishing sums of roots of unity. Berger's later theorem [1] implies the cotangent-ratio classification used here. We include a direct proof suited to the geometric question and do not claim a new trigonometric classification. Related commensurability questions for diagonals of regular polygons are studied in [10]; another setting in which rational-angle identities control regular polygons is the work of Connelly and Csikós on bicycle polygons [3].

Section 2 proves the polygon case and identifies the triangle–hexagon exception for rational area ratios. Section 3 proves the main theorem. We then extend the area argument to primitive regular star polygons, using their algebraic area. Appendix A treats numerical volume comparisons across different dimensions. All polytopes below are Euclidean, convex and regular unless a star polygon is explicitly specified. For a regular type  $P$ , the notation  $P(x)$  means a copy with edge length  $x > 0$ .

## 2. Polygons and cyclotomic fields

Write  $\zeta_n = e^{2\pi i/n}$ . We use the standard facts  $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$  and  $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ ; see [4, Theorems 2.5 and 4.1]. The field-coincidence argument needed here is included for completeness.

**Lemma 2.1** (Coincident cyclotomic fields; Conrad [4]). *For distinct integers  $n, m \geq 3$ ,*

$$\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_m) \iff \{n, m\} = \{k, 2k\} \text{ for an odd integer } k \geq 3$$

*Proof.* Put  $L = \text{lcm}(n, m)$ . Since  $\gcd(L/n, L/m) = 1$ , Bézout's identity shows that  $\zeta_L$  is a product of integer powers of  $\zeta_n$  and  $\zeta_m$ . Their compositum is therefore  $\mathbb{Q}(\zeta_L)$ . If the two fields are equal,

$$\varphi(n) = \varphi(m) = \varphi(L)$$

If  $a \mid b$ , each additional factor of an already occurring prime  $p$  multiplies the totient by  $p$ , whereas introducing a new prime power  $p^j$  multiplies it by  $p^{j-1}(p-1)$ . The only nontrivial change with multiplier 1 is introducing a single factor 2 into an odd integer. Hence  $a \mid b$  and  $\varphi(a) = \varphi(b)$  imply  $b = a$ , or  $b = 2a$  with  $a$  odd. Applying this to  $n, m \mid L$  proves the forward implication. Conversely, if  $k$  is odd, then  $\zeta_{2k} = -\zeta_k^{(k+1)/2}$ , so the fields agree.  $\square$

**Theorem 2.2** (Rational ratios at principal angles). *For integers  $n, m \geq 3$ ,*

$$\frac{\cot(\pi/n)}{\cot(\pi/m)} \in \mathbb{Q} \iff n = m \text{ or } \{n, m\} = \{3, 6\}$$

*Proof.* Set

$$u_n = \frac{\zeta_n + 1}{\zeta_n - 1} = -i \cot \frac{\pi}{n} \quad \zeta_n = \frac{u_n + 1}{u_n - 1}$$

The expression for  $u_n$  is the usual cyclotomic form of a cotangent number; see, for example, [7, Section 2]. Its inverse is the useful point here:  $\mathbb{Q}(u_n) = \mathbb{Q}(\zeta_n)$ . If  $u_n/u_m \in \mathbb{Q}^\times$ , then  $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_m)$ . For  $n \neq m$ , Lemma 2.1 lets us assume  $n = 2m$  with  $m$  odd. Now

$$\frac{\cot(\pi/(2m))}{\cot(\pi/m)} = 1 + \frac{1}{\cos(\pi/m)}$$

so  $2\cos(\pi/m)$  is rational. It is also an algebraic integer, being  $\zeta_{2m} + \zeta_{2m}^{-1}$ , and hence is an integer. Since  $1 \leq 2\cos(\pi/m) < 2$ , it equals 1, which forces  $m = 3$  and  $n = 6$ . Conversely, the exceptional cotangent ratios are 3 and  $1/3$ .  $\square$

**Theorem 2.3** (Area rigidity). *Let  $n, m \geq 3$  and  $x, y > 0$  with  $x/y \in \mathbb{Q}$ . Regular polygons with  $n, m$  sides and side lengths  $x, y$  have equal area if and only if  $n = m$  and  $x = y$ .*

*Proof.* Joining the centre to the vertices divides the polygon into  $n$  congruent triangles and gives the area formula

$$A_n(x) = \frac{nx^2}{4} \cot \frac{\pi}{n}$$

Equal areas imply a rational cotangent ratio. By Theorem 2.2, either  $n = m$ , giving  $x = y$ , or the polygons are a triangle and a hexagon. In the latter case, equality of  $A_3(x) = \sqrt{3}x^2/4$  and  $A_6(y) = 3\sqrt{3}y^2/2$  would give  $(x/y)^2 = 6$ , impossible for  $x/y \in \mathbb{Q}$ .  $\square$

**Corollary 2.4** (Rational area quotients). *For commensurable side lengths  $x, y > 0$ ,*

$$\frac{A_n(x)}{A_m(y)} \in \mathbb{Q} \iff n = m \text{ or } \{n, m\} = \{3, 6\}$$

*Thus the triangle and hexagon are the only distinct regular-polygon types whose areas can be commensurable when their edges are commensurable. This does not make equal areas possible: a unit-edge hexagon consists of six unit-edge equilateral triangles, and absorbing the factor 6 into an edge length requires multiplication by  $\sqrt{6}$ .*

*Proof.* The area quotient is  $\frac{n}{m}(x/y)^2 \cot(\pi/n)/\cot(\pi/m)$ ; apply Theorem 2.2.  $\square$

### 3. Regular polytopes in every dimension

For a fixed regular  $d$ -polytope type  $P$ , let  $\kappa(P)$  denote its volume at edge length 1. Homogeneity gives

$$\text{Vol}_d(P(x)) = \kappa(P)x^d \quad (1)$$

The geometric comparison therefore becomes a question about rational powers. Equal volumes with commensurable edges require  $\kappa(P)/\kappa(Q) = q^d$  for some  $q \in \mathbb{Q}_{>0}$ .

#### 3.1. One argument for the three infinite families

The regular simplex  $S_d$ , hypercube  $C_d$  and cross-polytope  $O_d$  have unit-edge volumes

$$\kappa(S_d) = \frac{\sqrt{d+1}}{2^{d/2}d!}, \quad \kappa(C_d) = 1, \quad \kappa(O_d) = \frac{2^{d/2}}{d!} \quad (2)$$

For the simplex, the Gram matrix of its  $d$  unit edges from one vertex is  $(I_d + J_d)/2$ , where every entry of  $J_d$  is 1. Its determinant is  $(d+1)/2^d$ . Dividing the square root by  $d!$  proves the first formula. The cube formula is immediate. The body  $\text{conv}\{\pm e_1, \dots, \pm e_d\}$  consists of  $2^d$  simplices of volume  $1/d!$  and has edge length  $\sqrt{2}$ ; rescaling gives the cross-polytope formula.

For a nonzero rational number  $r$ , the valuation  $v_p(r)$  is the exponent of  $p$  in its prime factorization, with negative exponents allowed. If  $r$  is a rational  $d$ th power, every  $v_p(r)$  is divisible by  $d$ . We will use a prime whose exponent is too small for that to happen.

**Lemma 3.1** (A prime with a useful exponent). *For every  $d \geq 3$ , there is an odd prime  $p \leq d$  such that*

$$p \nmid d+1, \quad 0 < v_p(d!) < d$$

*Proof.* If  $d$  is odd, choose  $p \mid d$ ; if  $d$  is even, choose  $p \mid d-1$ . In either case  $p$  is odd and cannot divide  $d+1$ . Moreover,

$$0 < v_p(d!) = \sum_{j \geq 1} \left\lfloor \frac{d}{p^j} \right\rfloor < \frac{d}{p-1} \leq \frac{d}{2} < d \quad \square$$

**Proposition 3.2** (The classical families). *For  $d \geq 3$ , no quotient between two distinct coefficients in (2) is a  $d$ th power in  $\mathbb{Q}_{>0}$ .*

*Proof.* Choose  $p$  as in Lemma 3.1 and write  $e = v_p(d!)$ . The squares of the simplex and cross-polytope coefficients satisfy

$$\kappa(S_d)^2 = \frac{d+1}{2^d(d!)^2}, \quad \kappa(O_d)^2 = \frac{2^d}{(d!)^2}$$

Both have  $p$ -adic valuation  $-2e$ , which is nonzero and has absolute value less than  $2d$ . Neither square is a rational  $2d$ th power. This excludes both comparisons with the cube.

For the remaining pair, if

$$\frac{\kappa(S_d)}{\kappa(O_d)} = \frac{\sqrt{d+1}}{2^d} = q^d$$

then  $(2q)^{2d} = d+1$ . A rational number with an integral positive integer power is itself an integer, as is seen by reducing its numerator and denominator. Thus  $2q$  is an integer, contradicting  $1 < d+1 < 2^{2d}$ .  $\square$

### 3.2. The exceptional types in dimensions three and four

The classical classification gives five types in dimension 3, six in dimension 4, and only  $S_d, C_d, O_d$  for  $d \geq 5$ . Besides the coefficients in (2), the required unit-edge values are as follows; the classification and metric formulas are recorded in [6].

| Dimension | Type               | Unit-edge volume         |
|-----------|--------------------|--------------------------|
| 3         | Dodecahedron $D$   | $(15 + 7\sqrt{5})/4$     |
| 3         | Icosahedron $I$    | $5(3 + \sqrt{5})/12$     |
| 4         | 24-cell $P_{24}$   | 2                        |
| 4         | 120-cell $P_{120}$ | $15(105 + 47\sqrt{5})/4$ |
| 4         | 600-cell $P_{600}$ | $25(2 + \sqrt{5})/4$     |

**Proposition 3.3** (The exceptional comparisons). *In dimensions 3 and 4, no quotient of unit-edge volumes of distinct regular types is a rational  $d$ th power.*

*Proof.* The three classical families are covered by Proposition 3.2. Their squared coefficients are rational. Each of the four coefficients in the table involving  $\sqrt{5}$  has the form  $a + b\sqrt{5}$  with nonzero  $a, b \in \mathbb{Q}$ , so its square is irrational. Thus its quotient with a classical coefficient, or with the 24-cell coefficient, cannot be rational. Within each exceptional pair,

$$\frac{\kappa(D)}{\kappa(I)} = \frac{3}{2} + \frac{9}{10}\sqrt{5}, \quad \frac{\kappa(P_{120})}{\kappa(P_{600})} = 15 + \frac{33}{5}\sqrt{5}$$

which are irrational. Finally, in dimension 4 the 24-cell divided by the cube, cross-polytope and simplex has quotient, respectively,

$$2, \quad 12, \quad \frac{192}{\sqrt{5}}$$

The first two are not rational fourth powers and the last is irrational.  $\square$

*Proof of Theorem 1.1.* For  $d = 2$ , use Theorem 2.3. For  $d = 3, 4$ , use Proposition 3.3; for  $d \geq 5$ , use the classification and Proposition 3.2. These rule out distinct types by (1). For a fixed type, positive volume and  $x^d = y^d$  give  $x = y$ . The converse is immediate.  $\square$

## 4. Rational angles and primitive regular star polygons

The same argument reaches all acute rational multiples of  $\pi$ . The classification below is a consequence of Berger's theorem on rational dependence of trigonometric numbers [1, Theorem 1.1 and Remark 1.1]. We give a direct proof using the largest absolute value of the conjugates of a cotangent generator.

**Theorem 4.1** (Arbitrary acute rational angles). *Let  $0 < \alpha, \beta < \pi/2$  with  $\alpha/\pi, \beta/\pi \in \mathbb{Q}$ . Then*

$$\frac{\cot \alpha}{\cot \beta} \in \mathbb{Q} \iff \alpha = \beta \text{ or } \{\alpha, \beta\} = \{\pi/6, \pi/3\}$$

*The rational ratios are 1, 3 and  $1/3$ . The same statement holds for tangent ratios.*

*Proof.* Write  $\alpha = a\pi/n$  and  $\beta = b\pi/m$  in lowest terms. For an algebraic number  $z$ , let  $M(z)$  be the maximum absolute value of its  $\mathbb{Q}$ -conjugates. The number

$$u_{n,a} = \frac{\zeta_n^a + 1}{\zeta_n^a - 1} = -i \cot(a\pi/n)$$

generates  $\mathbb{Q}(\zeta_n)$ , and its conjugates are  $-i \cot(k\pi/n)$  for  $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ . Positivity and strict decrease of cotangent on  $(0, \pi/2)$  give

$$M(u_{n,a}) = \cot(\pi/n)$$

If  $u_{n,a} = q u_{m,b}$  for  $q \in \mathbb{Q}_{>0}$ , then rational scaling of conjugates gives

$$\cot(\pi/n) = q \cot(\pi/m)$$

Theorem 2.2 implies  $n = m$  or  $\{n, m\} = \{3, 6\}$ . If  $n = m$ , the last equality gives  $q = 1$ , hence  $\alpha = \beta$ . For denominator 3 or 6, the only reduced fraction in  $(0, 1/2)$  is, respectively,  $1/3$  or  $1/6$ . This proves necessity; sufficiency follows from the familiar values at  $\pi/6$  and  $\pi/3$ . Taking reciprocals gives the tangent assertion.  $\square$

The following familiar consequence is included to illustrate the angle condition. It is a standard rational-value result; see [9, 8] and the explicit characterization in [2, Table 1].

**Corollary 4.2** (Rational-angle right triangles). *A right triangle with commensurable legs and both acute angles rational multiples of  $\pi$  is isosceles.*

*Proof.* Its acute angle  $\alpha$  has rational tangent. Apply Theorem 4.1 to  $\tan \alpha / \tan(\pi/4)$ .  $\square$

A *primitive regular star polygon*  $\{n/k\}$  is the closed path obtained by successively advancing  $k$  vertices around a regular  $n$ -gon, where  $1 \leq k < n/2$  and  $\gcd(k, n) = 1$ . This includes the convex case  $k = 1$ . Its *algebraic area* is the signed shoelace area of the path; regions are counted with winding multiplicity. It is not the area of the union of the regions enclosed by the drawing.

**Theorem 4.3** (Area rigidity for primitive stars). *For primitive stars  $\{n/k\}$  and  $\{m/\ell\}$  with commensurable edge lengths  $x, y > 0$ , equality of algebraic areas holds if and only if*

$$n = m, \quad k = \ell, \quad x = y$$

*Proof.* For circumradius  $R$ , the edge length is  $x = 2R \sin(k\pi/n)$  and the shoelace formula gives

$$\mathcal{A}_{n,k}(x) = \frac{n}{2} R^2 \sin \frac{2k\pi}{n} = \frac{nx^2}{4} \cot \frac{k\pi}{n}$$

Equal areas imply a rational cotangent ratio. By Theorem 4.1, either  $k/n = \ell/m$  or the two angles are  $\pi/6, \pi/3$ . In the first case, reduced fractions give  $n = m$  and  $k = \ell$ , and then  $x = y$ . The exceptional case consists precisely of the convex triangle and hexagon and is excluded by Theorem 2.3.  $\square$

The winding convention matters. For example, the central pentagon in a pentagram is counted twice in the shoelace area. A statement about the ordinary area of its filled drawing would be a different problem.

#### 4.1. What the hypothesis contributes

The result compares the edge lengths of the two objects, so it is unchanged when the unit of length is changed. The edges themselves need not be rational numbers: only their ratio must be rational. Regularity is essential as well. Rectangles of side lengths 1, 6 and 2, 3 have equal areas and integral edges, but are not congruent.

Within the stated class, volume determines both the regular-polytope type and its scale. The proof explains the two different obstructions: cyclotomic fields restrict the polygon types, while prime exponents rule out the rational rescalings that the higher-dimensional volume formulas would require.

### A. Numerical volumes in different dimensions

When dimensions differ, the numerical values of volumes are compared after fixing a common length unit. Such a comparison is not an equality of physical quantities with the same units. The following arithmetic observation applies whenever two quantities scale as fixed powers. Its proof is a direct use of Bézout's identity and prime factorization. We record the complete statement to make these comparisons easy to use.

**Theorem A.1** (Equality under integral scaling). *Let  $d, e \geq 1$ , let  $A, B > 0$ , and put  $g = \gcd(d, e)$ . The following conditions are equivalent:*

- (i)  $Ax^d = By^e$  for some  $x, y \in \mathbb{Z}_{>0}$ .
- (ii)  $Ax^d = By^e$  for some  $x, y \in \mathbb{Q}_{>0}$ .
- (iii)  $A/B = c^g$  for some  $c \in \mathbb{Q}_{>0}$ .

Equivalently,  $A/B$  must be rational and

$$g \mid v_p(A/B) \quad \text{for every prime } p$$

*Proof.* Write  $d = gd'$  and  $e = ge'$  with  $\gcd(d', e') = 1$ . Clearly (i) implies (ii), while (ii) gives

$$\frac{A}{B} = \left( \frac{y^{e'}}{x^{d'}} \right)^g$$

proving (iii). Conversely, assume  $A/B = c^g$  and choose integers  $u, v$  with  $e'u - d'v = 1$ . Then  $x = c^v$ ,  $y = c^u$  satisfy  $y^e/x^d = c^g = A/B$ . They are positive rationals. Replacing them by

$$xt^{e'}, \quad yt^{d'}$$

preserves the equation and makes both integers if the positive integer  $t$  is divisible by their denominators. The valuation formulation is the usual characterization of rational  $g$ th powers.  $\square$

**Corollary A.2** (All positive integer solutions). *Assume the equivalent conditions of Theorem A.1. There is a unique pair  $x_0, y_0 \in \mathbb{Z}_{>0}$  such that every positive integer solution is*

$$x = x_0 t^{e/g}, \quad y = y_0 t^{d/g}, \quad t \in \mathbb{Z}_{>0}$$

*and no prime  $p$  satisfies both  $p^{e/g} \mid x_0$  and  $p^{d/g} \mid y_0$ .*

*Proof.* Put  $k_p = v_p(A/B)$ . At each prime the equation is

$$da_p - eb_p = -k_p, \quad a_p, b_p \geq 0$$

All integral solutions differ by multiples of  $(e/g, d/g)$ . Since  $g \mid k_p$ , there is a unique nonnegative solution  $(a_p^0, b_p^0)$  from which this vector cannot be subtracted while keeping both coordinates nonnegative. Every other nonnegative solution is  $(a_p^0, b_p^0) + j_p(e/g, d/g)$  with  $j_p \geq 0$ . If  $k_p = 0$ , the minimal pair is  $(0, 0)$ . Hence the finite products

$$x_0 = \prod_p p^{a_p^0}, \quad y_0 = \prod_p p^{b_p^0}$$

have the asserted property, with  $t = \prod_p p^{j_p}$ .  $\square$

**Corollary A.3** (Application to polytopes). *Let  $P, Q$  be regular polytopes of dimensions  $d, e$ , respectively. Positive integral edge lengths giving equal numerical volumes exist if and only if*

$$\frac{\kappa(P)}{\kappa(Q)} \in \{c^{\gcd(d,e)} : c \in \mathbb{Q}_{>0}\}$$

*When they exist, all edge pairs are given by Corollary A.2. If  $d, e$  are coprime, rationality of the coefficient ratio alone suffices.*

*Proof.* Apply Theorem A.1 with  $A = \kappa(P)$  and  $B = \kappa(Q)$ .  $\square$

**Example A.4** (A cube and a 16-cell). A three-dimensional cube and a four-dimensional cross-polytope have coefficients 1 and  $1/6$ . Their dimensions are coprime, and

$$x^3 = \frac{y^4}{6} \iff x = 6t^4, \quad y = 6t^3, \quad t \in \mathbb{Z}_{>0}$$

In particular, edge length 6 gives numerical volume 216 in both cases. The minimal exponents at  $p = 2, 3$  are  $(1, 1)$ , which verifies the displayed parameterization directly.

**Example A.5** (Cubes in any two dimensions). For  $A = B = 1$ , the minimal pair is  $(x_0, y_0) = (1, 1)$ . Thus all positive integer solutions of  $x^d = y^e$  are

$$x = t^{e/g}, \quad y = t^{d/g}, \quad g = \gcd(d, e), \quad t \in \mathbb{Z}_{>0}$$

### A.1. A squarefree test for the classical families

For a positive integer  $N$ , let  $\text{sf}(N)$  be its squarefree part, so  $N = \text{sf}(N)s^2$  for an integer  $s > 0$ . Associate the following squarefree integer to a classical regular polytope:

$$\Delta(S_d) = \text{sf}(2^d(d+1)), \quad \Delta(C_d) = 1, \quad \Delta(O_d) = \text{sf}(2^d) = \begin{cases} 1, & d \text{ even,} \\ 2, & d \text{ odd} \end{cases}$$

**Proposition A.6** (Deciding rational volume ratios). *Let  $P$  and  $Q$  each belong to a classical family, possibly in different dimensions. Then*

$$\frac{\kappa(P)}{\kappa(Q)} \in \mathbb{Q} \iff \Delta(P) = \Delta(Q)$$

*When the squarefree parts agree, equality of numerical volumes at positive integral edges is decided by the prime-valuation condition in Corollary A.3.*

*Proof.* Formula (2) writes every coefficient as a positive rational multiple of  $\sqrt{\Delta(P)}$ ; for the simplex one may use

$$\kappa(S_d) = \frac{\sqrt{2^d(d+1)}}{2^d d!}$$

The quotient of square roots of positive squarefree integers is rational exactly when those integers agree. Apply Corollary A.3 for the remaining condition.  $\square$

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