

Analysis and Geometry on Manifolds Exercises

Due Dates. This exercise sheet is due 03.11.2025, end of lecture.

1. Coordinates [10 points]

Let (x, y) denote the standard coordinates on $\mathbb{R}^2$ and define new coordinates by $u := x$, $v := y - x^2$.

- (a) Show that (u, v) are smooth global coordinates.
 (b) Show that

$$\left. \frac{\partial}{\partial u} \right|_{(1,1)} \neq \left. \frac{\partial}{\partial x} \right|_{(1,1)}.$$

(Even though $u = x$).

2. Tangent spaces of products [10 points]

Let M and N be manifolds and let π_M and π_N denote the canonical projections from $M \times N$ to M and N . Show that for $p = (p_1, p_2) \in M \times N$, the map

$$\begin{aligned} \Psi: T_p(M \times N) &\rightarrow T_{p_1}M \oplus T_{p_2}N \\ v &\mapsto (d_p\pi_M(v), d_p\pi_N(v)) \end{aligned}$$

is an isomorphism.

3. Tangent space of the circle [10 points]

- (a) Consider the standard inclusion $\iota: S^1 \rightarrow \mathbb{R}^2$. Let (x, y) be the standard coordinates on $\mathbb{R}^2$, and let ϕ be the coordinate on (suitable open subsets of) S^1 given by $x = \cos \phi$ and $y = \sin \phi$. Given a point $p \in S^1$, compute the vector

$$d\iota_p \left(\left. \frac{\partial}{\partial \phi} \right|_p \right) \in T_{\iota(p)}\mathbb{R}^2$$

in terms of $\left. \frac{\partial}{\partial x} \right|_{\iota(p)}$ and $\left. \frac{\partial}{\partial y} \right|_{\iota(p)}$ (with coefficients expressed as functions of x and y).

- (b) Let

$$TS^1 := \{(p, \xi) \mid p \in S^1, \xi \in T_p S^1\}.$$

Show that there is a bijective map

$$\phi: S^1 \times \mathbb{R} \rightarrow TS^1,$$

such that $\phi(p, t) = (p, \psi(p, t))$ for $(p, t) \in S^1 \times \mathbb{R}$, and such that for all $p \in S^1$,

$$\mathbb{R} \ni t \mapsto \psi(p, t) \in T_p S^1,$$

is a vector space isomorphism.

- (c) Conclude that TS^1 is a smooth manifold of dimension 2.

4. Cubic coordinates [10 points]

Let $\mathbb{R}$ be the standard real line with global coordinate chart $(\mathbb{R}, \text{Id})$ and coordinate x . Let $\tilde{\mathbb{R}}$ be the smooth manifold given by the global chart $(\tilde{\mathbb{R}}, \psi)$ with coordinate y , where $\psi(q) := q^3$, for any $q \in \tilde{\mathbb{R}}$. Consider the diffeomorphism $F: \mathbb{R} \rightarrow \tilde{\mathbb{R}}$ given by $F(p) := p^{\frac{1}{3}}$ for $p \in \mathbb{R}$.

- (a) Compute the Jacobian matrix of the coordinate representation $\hat{F}$ of F , and use this to show that

$$dF \left(\left. \frac{\partial}{\partial x} \right|_p \right) = \left. \frac{\partial}{\partial y} \right|_{F(p)}. \quad (1)$$

- (b) For any $f \in C^\infty(\widetilde{\mathbb{R}})$, explicitly compute $\frac{\partial}{\partial y} \Big|_q f$ as a function of q and $\frac{\partial}{\partial x} \Big|_p (f \circ F)$ as a function of $F(p)$ (express both in terms of the usual derivative f'), and use this to verify equation (1).

If you find a mistake or if a question is unclear, please send an email to Oliver Füst (ofuerst@math.uni-bonn.de).