

Exercise Sheet 3 - Manifolds

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Exercise 1. [Coordinates] Let (x, y) denote the standard coordinates on $\mathbb{R}^2$ and define new coordinates by $u := x, v := y - x^2$.

- Show that (u, v) are smooth global coordinates.
- Show that

$$\left. \frac{\partial}{\partial u} \right|_{(1,1)} \neq \left. \frac{\partial}{\partial x} \right|_{(1,1)}$$

(Even though $u = x$).

Solution. Define $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\Phi(x, y) = (u, v) = (x, y - x^2).$$

Then Φ is smooth and has the smooth inverse

$$\Phi^{-1}(u, v) = (x, y) = (u, v + u^2).$$

Hence (u, v) are smooth global coordinates. For the second part, express the (u, v) -coordinate vector fields in terms of (x, y) : since $x = u$ and $y = v + u^2$, the chain rule gives

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + 2u \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v} \frac{\partial}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial}{\partial y} = \frac{\partial}{\partial y}.$$

At the point $(x, y) = (1, 1)$ we have $(u, v) = (1, 0)$, so

$$\left. \frac{\partial}{\partial u} \right|_{(1,1)} = \left(\frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y} \right) \Big|_{(1,1)} \neq \left. \frac{\partial}{\partial x} \right|_{(1,1)}.$$

Exercise 2. [Tangent spaces of products] Let M and N be manifolds and let π_M and π_N denote the canonical projections from $M \times N$ to M and N . Show that for $p = (p_1, p_2) \in M \times N$, the map

$$\begin{aligned} \Psi : T_p(M \times N) &\rightarrow T_{p_1}M \oplus T_{p_2}N \\ v &\rightarrow (d_p\pi_M(v), d_p\pi_N(v)) \end{aligned}$$

is an isomorphism.

Solution. Let M and N be smooth manifolds and $p = (p_1, p_2) \in M \times N$. Choose charts

$(M \supset U, x = (x^1, \dots, x^m))$ around p_1 , $(N \supset V, y = (y^1, \dots, y^n))$ around p_2 ,

and the product chart on $U \times V \subset M \times N$,

$$(x, y) : U \times V \longrightarrow \mathbb{R}^{m+n}, \quad (q_1, q_2) \mapsto (x(q_1), y(q_2)).$$

In these coordinates the projections are simply

$$\pi_M(q_1, q_2) = q_1 \quad \rightsquigarrow \quad x \circ \pi_M(x, y)^{-1}(u, v) = u$$

$$\pi_N(q_1, q_2) = q_2 \quad \rightsquigarrow \quad y \circ \pi_N(x, y)^{-1}(u, v) = v.$$

Hence, at p , the differentials in these coordinates are the canonical coordinate projections:

$$d_p \pi_M \cong \begin{pmatrix} I_m & 0 \end{pmatrix}, \quad d_p \pi_N \cong \begin{pmatrix} 0 & I_n \end{pmatrix}.$$

If we identify $T_p(M \times N)$ with $\mathbb{R}^{m+n}$ via the basis $\left\{ \frac{\partial}{\partial x^i} \Big|_p, \frac{\partial}{\partial y^j} \Big|_p \right\}$, and $T_{p_1}M \oplus T_{p_2}N$ with $\mathbb{R}^m \oplus \mathbb{R}^n$, then the map

$$\Psi : T_p(M \times N) \rightarrow T_{p_1}M \oplus T_{p_2}N, \quad v \mapsto (d_p \pi_M(v), d_p \pi_N(v))$$

is represented by the block matrix

$$\begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix},$$

which is the identity on $\mathbb{R}^{m+n}$. Therefore Ψ is a linear isomorphism.

Exercise 3. [Tangent space of the circle]

- Consider the standard inclusion $i : S^1 \rightarrow \mathbb{R}^2$. Let (x, y) be the standard coordinates on $\mathbb{R}^2$, and let ϕ be the coordinate on (suitable open subsets of) S^1 given by $x = \cos(\theta)$ and $y = \sin(\theta)$. Given a point $p \in S^1$, compute the vector

$$d_{i_p} \left(\frac{\partial}{\partial \phi} \Big|_p \right) \in T_{i(p)} \mathbb{R}^2$$

in terms of $\frac{\partial}{\partial x} \Big|_{i(p)}$ and $\frac{\partial}{\partial y} \Big|_{i(p)}$ (with coefficients expressed as functions of x and y).