



Ejemplos Resueltos del libro 5000 Problemas de Análisis Matemático (Demidovich)

Por

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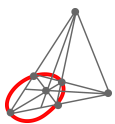
Escuela Superior de Física y Matemáticas

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Cálculo II

**Ejemplos Resueltos del libro 5000
Problemas de Análisis Matemático
(Demidovich)**

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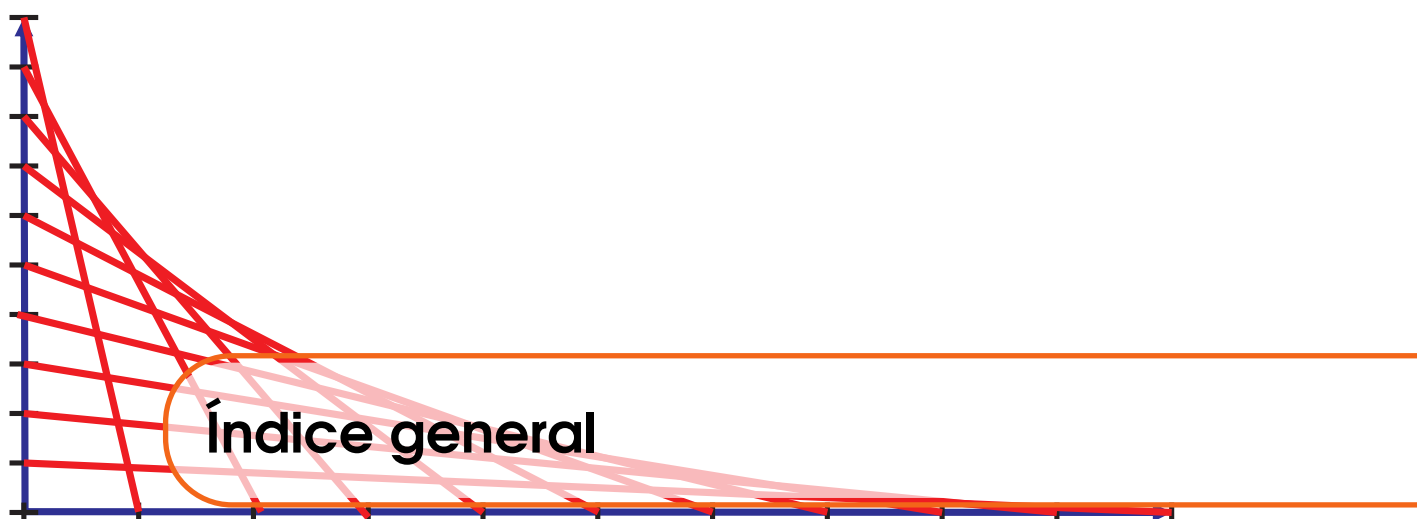
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La cultura y conocimiento son patrimonios de la humanidad. Comparte este conocimiento

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1	Integrales Indefinidas	7
1.1	Integrales Inmediatas	7
1.2	Cambio de variable	17
1.3	Integración de funciones trigonométricas 1ª parte	54
1.4	Cambio de variable y sustitución trigonométrica	62
2	Diferentes Técnicas de Integración	73
2.1	Integración por partes	73
2.2	Integrales que contienen un trinomio cuadrado	99
2.3	Integración de funciones racionales	129
2.4	Integración de funciones irracionales	164
2.5	Integración de Funciones Trigonómicas 2ª parte	184
3	Integrales Definidas	223
3.1	Cálculo de integrales definidas mediante integrales indefinidas	223
3.2	Integrales Impropias	241

1 — Integrales Indefinidas

1.1 Integrales Inmediatas

Ejemplo 1.1

$$1628 \int (3 - x^2)^3 dx$$

Solución

Desarrollando:

$$(3 - x^2)^3 = -x^6 + 9x^4 - 27x^2 + 27,$$

por lo cual:

$$\begin{aligned} \int (3 - x^2)^3 dx &= \int (-x^6 + 9x^4 - 27x^2 + 27) dx \\ &= -\int x^6 dx + 9 \int x^4 dx - 27 \int x^2 dx + 27 \int dx \\ &= -\frac{1}{7}x^7 + \frac{9}{5}x^5 - 9x^3 + 27x + c \end{aligned}$$

$$\text{RESPUESTA: } -\frac{1}{7}x^7 + \frac{9}{5}x^5 - 9x^3 + 27x + c$$

Ejemplo 1.2

$$1630 \int (1-x)(1-2x)(1-3x) dx$$

Solución

Desarrollando:

$$(1-x)(1-2x)(1-3x) = -6x^3 + 11x^2 - 6x + 1,$$

por lo cual:

$$\begin{aligned} \int (1-x)(1-2x)(1-3x) dx &= \int (-6x^3 + 11x^2 - 6x + 1) dx \\ &= -6 \int x^3 dx + 11 \int x^2 dx - 6 \int x dx + \int dx \\ &= -\frac{3}{2}x^4 + \frac{11}{3}x^3 - 3x^2 + x + c \end{aligned}$$

$$\text{RESPUESTA: } -\frac{3}{2}x^4 + \frac{11}{3}x^3 - 3x^2 + x + c$$

Ejemplo 1.3

$$1632 \int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3} \right) dx$$

Solución

$$\begin{aligned} \int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3} \right) dx &= a \int \frac{dx}{x} + a^2 \int \frac{dx}{x^2} + a^3 \int \frac{dx}{x^3} \\ &= a \ln x - \frac{a^2}{x} - \frac{a^3}{2x^2} + c \end{aligned}$$

$$\text{RESPUESTA: } a \ln x - \frac{a^2}{x} - \frac{a^3}{2x^2} + c$$

Ejemplo 1.4

$$1634 \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} - 1}{\sqrt[4]{x}} dx$$

Solución

Desarrollando:

$$\begin{aligned} \frac{\sqrt{x} - 2\sqrt[3]{x^2} - 1}{\sqrt[4]{x}} &= \frac{x^{\frac{1}{2}} - 2x^{\frac{2}{3}} - 1}{x^{\frac{1}{4}}} \\ &= x^{\frac{1}{2} - \frac{1}{4}} - 2x^{\frac{2}{3} - \frac{1}{4}} - x^{-\frac{1}{4}} \\ &= x^{\frac{1}{4}} - 2x^{\frac{5}{12}} - x^{-\frac{1}{4}}, \end{aligned}$$

por lo cual:

$$\begin{aligned} \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} - 1}{\sqrt[4]{x}} dx &= \int \left(x^{\frac{1}{4}} - 2x^{\frac{5}{12}} - x^{-\frac{1}{4}} \right) dx \\ &= \int x^{\frac{1}{4}} dx - 2 \int x^{\frac{5}{12}} dx - \int x^{-\frac{1}{4}} dx \\ &= \frac{4}{5} x^{\frac{5}{4}} - \frac{24}{17} x^{\frac{17}{12}} - \frac{4}{3} x^{\frac{3}{4}} + c \end{aligned}$$

RESPUESTA: $\frac{4}{5} x^{\frac{5}{4}} - \frac{24}{17} x^{\frac{17}{12}} - \frac{4}{3} x^{\frac{3}{4}} + c$

Ejemplo 1.5

$$1636 \int \left(1 - \frac{1}{x^2} \right) \sqrt{x\sqrt{x}} dx$$

Solución

Desarrollando:

$$\begin{aligned} \sqrt{x\sqrt{x}} &= \left(x \left(x^{\frac{1}{2}} \right) \right)^{\frac{1}{2}}, \\ &= \left(x^{\frac{3}{2}} \right)^{\frac{1}{2}}, \\ &= x^{\frac{3}{4}}, \end{aligned}$$

multiplicando por $\left(1 - \frac{1}{x^2}\right)$:

$$\begin{aligned}\left(1 - \frac{1}{x^2}\right) \sqrt{x\sqrt{x}} &= \left(1 - \frac{1}{x^2}\right) x^{\frac{3}{4}}, \\ &= x^{\frac{3}{4}} - \frac{x^{\frac{3}{4}}}{x^2}, \\ &= x^{\frac{3}{4}} - x^{-\frac{5}{4}},\end{aligned}$$

por lo cual:

$$\begin{aligned}\int \left(1 - \frac{1}{x^2}\right) \sqrt{x\sqrt{x}} dx &= \int \left(x^{\frac{3}{4}} - x^{-\frac{5}{4}}\right) dx \\ &= \int x^{\frac{3}{4}} dx - \int x^{-\frac{5}{4}} dx \\ &= \frac{4}{7} x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + c\end{aligned}$$

RESPUESTA: $\frac{4}{7} x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + c$

Ejemplo 1.6

$$1638 \int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx$$

Solución

Factorizando:

$$\begin{aligned}(x^4 + x^{-4} + 2) &= (x^2)^2 + 2(x^2)(x^{-2}) + (x^{-2})^2 \\ &= (x^2 + x^{-2})^2,\end{aligned}$$

de donde se obtiene:

$$\begin{aligned}\sqrt{x^4 + x^{-4} + 2} &= \sqrt{(x^2 + x^{-2})^2} \\ &= x^2 + x^{-2},\end{aligned}$$

luego, al dividir por x^3 :

$$\frac{x^2 + x^{-2}}{x^3} = \frac{1}{x} + \frac{1}{x^5},$$

por lo cual:

$$\begin{aligned}\int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx &= \int \left(\frac{1}{x} + \frac{1}{x^5} \right) dx \\ &= \int \frac{dx}{x} + \int \frac{dx}{x^5} \\ &= \ln x - \frac{1}{4x^4} + c\end{aligned}$$

RESPUESTA: $\ln x - \frac{1}{4x^4} + c$

Ejemplo 1.7

1640 $\int \frac{x^2}{1-x^2} dx$

Solución

Desarrollando:

$$\begin{aligned}\frac{x^2}{1-x^2} &= \frac{x^2 - 1 + 1}{1-x^2} \\ &= \frac{x^2 - 1}{1-x^2} + \frac{1}{1-x^2} \\ &= -1 + \frac{1}{1-x^2},\end{aligned}$$

por lo cual:

$$\begin{aligned}\int \frac{x^2}{1-x^2} dx &= \int \left(-1 + \frac{1}{1-x^2} \right) dx \\ &= -\int dx + \int \frac{dx}{1-x^2} \\ &= -x + \frac{1}{2} \ln \left(\frac{x+1}{1-x} \right) + c\end{aligned}$$

RESPUESTA: $-x + \frac{1}{2} \ln \left(\frac{x+1}{1-x} \right) + c$

Ejemplo 1.8

$$1642 \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx$$

Solución

Desarrollando:

$$\begin{aligned} \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} &= \frac{\sqrt{1+x^2}}{\sqrt{1-x^4}} + \frac{\sqrt{1-x^2}}{\sqrt{1-x^4}} \\ &= \frac{\sqrt{1+x^2}}{\sqrt{(1+x^2)(1-x^2)}} + \frac{\sqrt{1-x^2}}{\sqrt{(1+x^2)(1-x^2)}} \\ &= \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}, \end{aligned}$$

por lo cual:

$$\begin{aligned} \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx &= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}} \right) dx \\ &= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1+x^2}} \\ &= \arcsin x + \ln \left(x + \sqrt{x^2 + 1} \right) + c \end{aligned}$$

RESPUESTA: $\arcsin x + \ln \left(x + \sqrt{x^2 + 1} \right) + c$

Ejemplo 1.9

$$1644 \int (2^x + 3^x)^2 dx$$

Solución

Desarrollando:

$$\begin{aligned} (2^x + 3^x)^2 &= 2^{2x} + 3^{2x} + 2(2^x 3^x) \\ &= (2^2)^x + (3^2)^x + 2(6)^x \\ &= 4^x + 9^x + 2(6^x), \end{aligned}$$

por lo cual:

$$\begin{aligned}\int (2^x + 3^x) dx &= \int (4^x + 9^x + 2(6^x)) dx \\ &= \int 4^x dx + \int 9^x dx + 2 \int 6^x dx \\ &= \frac{4^x}{\ln 4} + \frac{9^x}{\ln 9} + 2 \frac{6^x}{\ln 6} + c\end{aligned}$$

RESPUESTA: $\frac{4^x}{\ln 4} + \frac{9^x}{\ln 9} + 2 \frac{6^x}{\ln 6} + c$

Ejemplo 1.10

1646 $\int \frac{e^{2x} + 1}{e^x + 1} dx$

Solución

Desarrollando:

$$\begin{aligned}\frac{e^{2x} + 1}{e^x + 1} &= \frac{(e^x)^2 + 2e^x + 1 - 2e^x}{e^x + 1} \\ &= \frac{((e^x)^2 + 2e^x + 1)}{e^x + 1} - 2 \frac{e^x}{e^x + 1} \\ &= \frac{(e^x + 1)^2}{e^x + 1} - 2 \frac{e^x}{e^x + 1} \\ &= e^x + 1 - 2 \frac{e^x}{e^x + 1},\end{aligned}$$

por lo cual:

$$\begin{aligned}\int \frac{e^{2x} + 1}{e^x + 1} dx &= \int \left(e^x + 1 - 2 \frac{e^x}{e^x + 1} \right) dx \\ &= \int e^x dx + \int dx - 2 \int \frac{e^x}{e^x + 1} dx,\end{aligned}$$

ahora consideremos el cambio de variable:

$$t = 1 + e^x$$

$$dt = e^x dx$$

sustituyendo en $\int \frac{e^x}{e^x + 1} dx$:

$$\begin{aligned}\int \frac{e^x}{1 + e^x} dx &= \int \frac{dt}{t} \\ &= \ln t \\ &= \ln(e^x + 1),\end{aligned}$$

finalmente, por todo lo anterior:

$$\begin{aligned}\int \frac{e^{2x}+1}{e^x+1} dx &= \int e^x dx + \int dx - 2 \int \frac{e^x}{e^x+1} dx \\ &= e^x + x - 2 \ln(e^x + 1) + c\end{aligned}$$

RESPUESTA: $e^x + x - 2 \ln(e^x + 1) + c$

Ejemplo 1.11

1648 $\int \sqrt{1 - \sin(2x)} dx, (0 \leq x \leq \pi)$

Solución

El problema se puede resolver considerando diferentes identidades trigonométricas:

Caso a)

Desarrollando:

$$\begin{aligned}1 - \sin(2x) &= \cos^2 x - 2 \sin x \cos x + \sin^2 x \\ &= (\cos x - \sin x)^2\end{aligned}$$

Sin embargo, conviene trabajar mejor con el caso que se muestra a continuación.

Caso b)

Desarrollando:

$$\begin{aligned}\sin(2x) &= 2 \sin x \cos x \\ &= 2 \frac{\sin x}{\cos x} \cos^2 x \\ &= \frac{2 \tan x}{\sec^2 x} \\ &= \frac{2 \tan x}{1 + \tan^2 x},\end{aligned}$$

ahora simplificaremos:

$$\begin{aligned}1 - \sin(2x) &= 1 - \frac{2 \tan x}{1 + \tan^2 x} \\ &= \frac{1 - 2 \tan x + \tan^2 x}{1 + \tan^2 x} \\ &= \frac{(1 - \tan x)^2}{1 + \tan^2 x} \\ &= \frac{(1 - \tan x)^2}{\sec^2 x} \\ &= \frac{(\tan x - 1)^2}{\sec^2 x},\end{aligned}$$

en $x = \frac{\pi}{4}$ la función alcanza su valor mínimo, por lo que conviene trabajar en dos subintervalos. Primero consideremos $0 \leq x \leq \frac{\pi}{4}$:

$$\begin{aligned}\sqrt{1 - \sin 2x} &= \frac{1 - \tan x}{\sec x} \\ &= \cos x - \sin x,\end{aligned}$$

para $\frac{\pi}{4} \leq x \leq \pi$:

$$\begin{aligned}\sqrt{1 - \sin 2x} &= \frac{\tan x - 1}{\sec x} \\ &= \sin x - \cos x,\end{aligned}$$

por lo anterior:

$$\begin{aligned}\int \sqrt{1 - \sin 2x} dx &= \int_{0 \leq x \leq \frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4} \leq x \leq \pi} (\sin x - \cos x) dx \\ &= \begin{cases} \sin x + \cos x + c, & \text{para } 0 \leq x \leq \frac{\pi}{4}, \\ -(\sin x + \cos x) + c, & \text{para } \frac{\pi}{4} \leq x \leq \pi \end{cases}\end{aligned}$$

RESPUESTA:

$$\begin{cases} \sin x + \cos x + c, & \text{para } 0 \leq x \leq \frac{\pi}{4}, \\ -(\sin x + \cos x) + c, & \text{para } \frac{\pi}{4} \leq x \leq \pi \end{cases}$$

Ejemplo 1.12

$$1650 \int \tan^2 x dx$$

Solución

Consideremos la siguiente identidad:

$$\begin{aligned}1 + \tan^2 x &= \sec^2 x \\ \tan^2 x &= \sec^2 x - 1,\end{aligned}$$

por lo anterior:

$$\begin{aligned}\int \tan^2 x dx &= \int (\sec^2 x - 1) dx \\ &= \int \sec^2 x dx - \int dx \\ &= \tan x - x + c\end{aligned}$$

RESPUESTA: $\tan x - x + c$

1.2 Cambio de variable

Ejemplo 1.13

1654 Demostrar que, si

$$\int f(x) dx = F(x) + c,$$

entonces

$$\int f(ax+b) dx = \frac{1}{a} F(ax+b) + c, \quad a \neq 0.$$

Demostración

Consideremos $t = ax + b$:

$$\begin{aligned} t &= ax + b, \\ dt &= a dx, \\ \frac{dt}{a} &= dx, \end{aligned}$$

sustituyendo en $\int f(ax+b) dx$:

$$\begin{aligned} \int f(ax+b) dx &= \int f(t) \left(\frac{dt}{a} \right) \\ &= \frac{1}{a} \int f(t) dt \\ &= \frac{1}{a} F(t) + c \\ &= \frac{1}{a} F(ax+b) + c \quad \blacksquare \end{aligned}$$

Ejemplo 1.14

1656 $\int (2x-3)^{10} dx$

Solución

Consideremos $t = 2x - 3$:

$$\begin{aligned} t &= 2x - 3, \\ dt &= 2 dx, \\ \frac{dt}{2} &= dx, \end{aligned}$$

sustituyendo en $\int (2x-3)^{10} dx$:

$$\begin{aligned}\int (2x-3)^{10} dx &= \int t^{10} \left(\frac{dt}{2} \right) \\&= \frac{1}{2} \int t^{10} dt \\&= \frac{1}{2} \left(\frac{1}{11} \right) t^{11} + c \\&= \frac{1}{22} (2x-3)^{11} + c\end{aligned}$$

RESPUESTA: $\frac{1}{22} (2x-3)^{11} + c$

Ejemplo 1.15

1658 $\int \frac{dx}{\sqrt{2-5x}}$

Solución

Consideremos $t = 2 - 5x$:

$$\begin{aligned}t &= 2 - 5x, \\dt &= -5dx, \\-\frac{dt}{5} &= dx,\end{aligned}$$

sustituyendo en $\int \frac{dx}{\sqrt{2-5x}}$:

$$\begin{aligned}\int \frac{dx}{\sqrt{2-5x}} &= \int \frac{1}{\sqrt{t}} \left(\frac{dt}{-5} \right) \\&= -\frac{1}{5} \int \frac{dt}{\sqrt{t}} \\&= -\frac{1}{5} \int t^{-\frac{1}{2}} dt \\&= -\frac{2}{5} t^{\frac{1}{2}} + c \\&= -\frac{2}{5} \sqrt{2-5x} + c\end{aligned}$$

RESPUESTA: $-\frac{2}{5}\sqrt{2-5x}+c$

Ejemplo 1.16

1660 $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x} dx$

Solución

Simplificando:

$$\begin{aligned}\frac{\sqrt[5]{1-2x+x^2}}{1-x} &= \frac{\left((1-x)^2\right)^{\frac{1}{5}}}{1-x} \\ &= \frac{(1-x)^{\frac{2}{5}}}{1-x} \\ &= (1-x)^{\frac{2}{5}-1} \\ &= (1-x)^{-\frac{3}{5}},\end{aligned}$$

de lo anterior:

$$\int \frac{\sqrt[5]{1-2x+x^2}}{1-x} dx = \int (1-x)^{-\frac{3}{5}} dx,$$

consideremos $t = 1-x$:

$$\begin{aligned}t &= 1-x, \\ dt &= -dx, \\ -dt &= dx,\end{aligned}$$

sustituyendo en $\int (1-x)^{-\frac{3}{5}} dx$:

$$\begin{aligned}\int (1-x)^{-\frac{3}{5}} dx &= \int t^{-\frac{3}{5}} (-dt), \\ &= -\int t^{-\frac{3}{5}} dt, \\ &= -\frac{5}{2} t^{\frac{2}{5}} + c, \\ &= -\frac{5}{2} (1-x)^{\frac{2}{5}} + c\end{aligned}$$

RESPUESTA: $-\frac{5}{2}(1-x)^{\frac{2}{5}} + c$

Ejemplo 1.17

1662 $\int \frac{dx}{2-3x^2}$

Solución

Desarrollando:

$$\frac{1}{2-3x^2} = \frac{1}{2} \left(\frac{1}{1-\frac{3x^2}{2}} \right),$$

consideremos $t = \sqrt{\frac{3}{2}}x$:

$$t = \sqrt{\frac{3}{2}}x,$$

$$dt = \sqrt{\frac{3}{2}}dx,$$

$$\sqrt{\frac{2}{3}}dt = dx,$$

sustituyendo en $\int \frac{dx}{2-3x^2}$:

$$\begin{aligned} \int \frac{dx}{2-3x^2} &= \frac{1}{2} \int \frac{1}{1-t^2} \left(\sqrt{\frac{2}{3}} dt \right) \\ &= \frac{\sqrt{2}}{2\sqrt{3}} \int \frac{dt}{1-t^2} \\ &= \left(\frac{\sqrt{2}}{2\sqrt{3}} \right) \left(\frac{1}{2} \right) \ln \left(\frac{1+t}{1-t} \right) + c \\ &= \frac{\sqrt{6}}{12} \ln \left(\frac{1+\sqrt{\frac{3}{2}}x}{1-\sqrt{\frac{3}{2}}x} \right) + c \\ &= \frac{\sqrt{6}}{12} \ln \left(\frac{\sqrt{6}+3x}{\sqrt{6}-3x} \right) + c \end{aligned}$$

RESPUESTA: $\frac{\sqrt{6}}{12} \ln \left(\frac{\sqrt{6}+3}{\sqrt{6}-3x} \right) + c$

Ejemplo 1.18

1664 $\int \frac{dx}{\sqrt{3x^2-2}}$

Solución

Desarrollando:

$$\begin{aligned} \frac{1}{\sqrt{3x^2-2}} &= \frac{1}{\sqrt{3\left(x^2-\frac{2}{3}\right)}}, \\ &= \frac{1}{\sqrt{3}\sqrt{x^2-\frac{2}{3}}}, \\ &= \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{x^2-\frac{2}{3}}} \right), \end{aligned}$$

por lo anterior:

$$\int \frac{dx}{\sqrt{3x^2-2}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2-\frac{2}{3}}}$$

aplicando la regla $\int \frac{du}{\sqrt{u^2-a^2}} = \ln(u + \sqrt{u^2-a^2})$ con $a = \sqrt{\frac{2}{3}}$:

$$\begin{aligned} \int \frac{dx}{\sqrt{3x^2-2}} &= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2-\frac{2}{3}}}, \\ &= \frac{1}{\sqrt{3}} \ln \left(x + \sqrt{x^2-\frac{2}{3}} \right) + c \end{aligned}$$

RESPUESTA: $\frac{1}{\sqrt{3}} \ln \left(x + \sqrt{x^2-\frac{2}{3}} \right) + c$

Ejemplo 1.19

$$1668 \int \frac{dx}{1 + \cos x}$$

Solución

Desarrollando:

$$\begin{aligned} \frac{1}{1 + \cos x} &= \frac{1}{1 + \cos x} \left(\frac{1 - \cos x}{1 - \cos x} \right) \\ &= \frac{1 - \cos x}{1 - \cos^2 x} \\ &= \frac{1 - \cos x}{\sin^2 x} \\ &= \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}, \end{aligned}$$

de lo anterior tenemos:

$$\int \frac{dx}{1 + \cos x} = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x dx}{\sin^2 x},$$

ahora calcularemos $\int \frac{\cos x}{\sin^2 x} dx$, para ello consideremos $t = \sin x$:

$$t = \sin x,$$

$$dt = \cos x dx,$$

sustituyendo en $\int \frac{\cos x}{\sin^2 x} dx$:

$$\begin{aligned} \int \frac{\cos x}{\sin^2 x} dx &= \int \frac{dt}{t^2} \\ &= \int t^{-2} dt \\ &= -\frac{1}{t} \\ &= -\frac{1}{\sin x}, \end{aligned}$$

por lo cual:

$$\begin{aligned} \int \frac{dx}{1 + \cos x} &= \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x} dx \\ &= -\cot x - \left(-\frac{1}{\sin x} \right) + c \\ &= -\frac{\cos x}{\sin x} + \frac{1}{\sin x} + c \\ &= \frac{1 - \cos x}{\sin x} + c \\ &= \frac{\sin x}{1 + \cos x} + c \end{aligned}$$

RESPUESTA: $\frac{\sin x}{1 + \cos x} + c$

Ejemplo 1.20

1670 $\int \frac{dx}{1 + \sin x}$

Solución

Desarrollando:

$$\begin{aligned} \frac{1}{1 + \sin x} &= \frac{1}{1 + \sin x} \left(\frac{1 - \sin x}{1 - \sin x} \right) \\ &= \frac{1 - \sin x}{1 - \sin^2 x} \\ &= \frac{1 - \sin x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}, \end{aligned}$$

de lo anterior tenemos:

$$\int \frac{dx}{1 + \sin x} = \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x} dx,$$

ahora calcularemos $\int \frac{\sin x}{\cos^2 x} dx$, para ello consideremos $t = \cos x$:

$$\begin{aligned} t &= \cos x, \\ dt &= -\sin x dx, \\ -dt &= \sin x dx \end{aligned}$$

sustituyendo en $\int \frac{\sin x}{\cos^2 x} dx$:

$$\begin{aligned} \int \frac{\sin x}{\cos^2 x} dx &= \int -\frac{dt}{t^2} \\ &= \frac{1}{t} \\ &= \frac{1}{\cos x}, \end{aligned}$$

por lo cual:

$$\begin{aligned}
 \int \frac{dx}{1 + \sin x} &= \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x} dx \\
 &= \tan x - \frac{1}{\cos x} + c \\
 &= \frac{\sin x}{\cos x} - \frac{1}{\cos x} + c \\
 &= \frac{\sin x - 1}{\cos x} + c \\
 &= -\frac{\cos x}{1 + \sin x} + c
 \end{aligned}$$

RESPUESTA: $-\frac{\cos x}{1 + \sin x} + c$

Ejemplo 1.21

1674 $\int \frac{x}{\sqrt{1-x^2}} dx$

Solución

Considerando $t = 1 - x^2$:

$$\begin{aligned}
 t &= 1 - x^2, \\
 dt &= -2x dx, \\
 -\frac{dt}{2} &= x dx,
 \end{aligned}$$

sustituyendo en $\int \frac{x}{\sqrt{1-x^2}} dx$:

$$\begin{aligned}
 \int \frac{x}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{t}} \left(\frac{-dt}{2} \right) \\
 &= -\frac{1}{2} \int \frac{dt}{\sqrt{t}} \\
 &= -\frac{1}{2} (2\sqrt{t}) \\
 &= -\sqrt{1-x^2} + c
 \end{aligned}$$

RESPUESTA: $-\sqrt{1-x^2} + c$

Ejemplo 1.22

$$1676 \int \frac{x}{3-2x^2} dx$$

Solución

Considerando $t = 3 - 2x^2$:

$$\begin{aligned} t &= 3 - 2x^2, \\ dt &= -4x dx, \\ -\frac{dt}{4} &= x dx, \end{aligned}$$

sustituyendo en $\int \frac{x}{3-2x^2} dx$:

$$\begin{aligned} \int \frac{x}{3-2x^2} dx &= \int \frac{1}{t} \left(-\frac{dt}{4} \right) \\ &= -\frac{1}{4} \int \frac{dt}{t} \\ &= -\frac{1}{4} \ln t + c \\ &= -\frac{1}{4} \ln (3 - 2x^2) + c \end{aligned}$$

RESPUESTA: $-\frac{1}{4} \ln (3 - 2x^2) + c$

Ejemplo 1.23

$$1678 \int \frac{x}{4+x^4} dx$$

Solución

Considerando $t = x^2$:

$$\begin{aligned} t &= x^2, \\ dt &= 2x dx, \\ \frac{dt}{2} &= x dx, \end{aligned}$$

sustituyendo en $\int \frac{x}{4+x^4} dx$:

$$\begin{aligned}\int \frac{x}{4+(x^2)^2} dx &= \int \frac{1}{4+t^2} \left(\frac{dt}{2} \right) \\ &= \frac{1}{2} \int \frac{dt}{4+t^2},\end{aligned}$$

aplicando la regla $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a}$ con $a = 2$:

$$\begin{aligned}\int \frac{x}{4+x^4} dx &= \frac{1}{2} \int \frac{dt}{4+t^2} \\ &= \frac{1}{2} \left(\frac{1}{2} \arctan \left(\frac{t}{2} \right) \right) + c \\ &= \frac{1}{4} \arctan \left(\frac{x^2}{2} \right) + c\end{aligned}$$

RESPUESTA: $\frac{1}{4} \arctan \left(\frac{x^2}{2} \right) + c$

Ejemplo 1.24

1680 $\int \frac{dx}{(1+x)\sqrt{x}}$

Solución

Consideremos $t^2 = x$:

$$\begin{aligned}t^2 &= x, \\ 2t dt &= dx, \\ t &= \sqrt{x},\end{aligned}$$

sustituyendo en $\int \frac{dx}{(1+x)\sqrt{x}}$:

$$\begin{aligned}\int \frac{dx}{(1+x)\sqrt{x}} &= \int \frac{2t}{t(1+t^2)} dt \\ &= 2 \int \frac{dt}{1+t^2} \\ &= 2 \arctan t + c \\ &= 2 \arctan (\sqrt{x}) + c\end{aligned}$$

RESPUESTA: $2 \arctan(\sqrt{x}) + c$

Ejemplo 1.25

1682 $\int \frac{dx}{x\sqrt{x^2+1}}$

Solución a)

Consideremos $\frac{1}{t} = x$:

$$\begin{aligned}\frac{1}{t} &= x, \\ -\frac{dt}{t^2} &= dx,\end{aligned}$$

sustituyendo en $\frac{1}{x\sqrt{x^2+1}}$:

$$\begin{aligned}\frac{1}{x\sqrt{x^2+1}} &= \frac{1}{\frac{1}{t}\sqrt{\frac{1}{t^2}+1}} \\ &= \frac{1}{\frac{1}{t}\sqrt{\frac{1+t^2}{t^2}}} \\ &= \frac{1}{\frac{\sqrt{1+t^2}}{t^2}} \\ &= \frac{t^2}{\sqrt{1+t^2}},\end{aligned}$$

sustituyendo en $\int \frac{dx}{x\sqrt{x^2+1}}$:

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2+1}} &= \int \frac{t^2}{\sqrt{1+t^2}} \left(-\frac{dt}{t^2}\right) \\ &= -\int \frac{dt}{\sqrt{1+t^2}} \\ &= -\ln\left(t + \sqrt{t^2+1}\right) + c \\ &= -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + c.\end{aligned}$$

$$\text{RESPUESTA A): } -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) + c$$

Solución b); Cambio de variable sugerido por Ollin Reyes Zamora, Marzo, 2021

Consideremos $u = \sqrt{x^2 + 1}$:

$$\begin{aligned} u &= \sqrt{x^2 + 1}, \\ du &= \frac{x}{\sqrt{x^2 + 1}} dx, \\ u^2 &= x^2 + 1, \\ u^2 - 1 &= x^2, \end{aligned}$$

sustituyendo en $\int \frac{dx}{x\sqrt{x^2 + 1}}$:

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2 + 1}} &= \int \frac{1}{x^2} \left(\frac{x}{\sqrt{x^2 + 1}} \right) dx \\ &= \int \frac{du}{(u^2 - 1)} \\ &= \frac{1}{2} \ln(1 - u) - \frac{1}{2} \ln(u + 1) + c \\ &= \frac{1}{2} \ln\left(\frac{1 - \sqrt{x^2 + 1}}{1 + \sqrt{x^2 + 1}}\right) + c \end{aligned}$$

$$\text{RESPUESTA B): } \frac{1}{2} \ln\left(\frac{1 - \sqrt{x^2 + 1}}{1 + \sqrt{x^2 + 1}}\right) + c$$

Ejemplo 1.26

$$1684 \int \frac{dx}{(x^2 + 1)^{\frac{3}{2}}}$$

Solución a)

Desarrollando:

$$\begin{aligned}(x^2 + 1)^{\frac{3}{2}} &= \left(x^2 \left(1 + \frac{1}{x^2}\right)\right)^{\frac{3}{2}} \\ &= (x^2)^{\frac{3}{2}} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \\ &= x^3 \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}},\end{aligned}$$

de lo anterior:

$$\int \frac{dx}{(x^2 + 1)^{\frac{3}{2}}} = \int \frac{1}{\left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}}} \left(\frac{dx}{x^3}\right),$$

considerando $t = 1 + \frac{1}{x^2}$:

$$\begin{aligned}t &= 1 + \frac{1}{x^2}, \\ dt &= -\frac{2dx}{x^3}, \\ -\frac{dt}{2} &= \frac{dx}{x^3},\end{aligned}$$

sustituyendo en $\int \frac{1}{\left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}}} \left(\frac{dx}{x^3}\right)$:

$$\begin{aligned}\int \frac{1}{\left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}}} \left(\frac{dx}{x^3}\right) &= \int \frac{1}{t^{\frac{3}{2}}} \left(-\frac{dt}{2}\right) \\ &= -\frac{1}{2} \int \frac{dt}{t^{\frac{3}{2}}} \\ &= -\frac{1}{2} \left(\frac{-2}{\sqrt{t}}\right) + c \\ &= \frac{1}{\sqrt{t}} + c \\ &= \frac{1}{\sqrt{1 + \frac{1}{x^2}}} + c \\ &= \frac{x}{\sqrt{x^2 + 1}} + c\end{aligned}$$

RESPUESTA A): $\frac{x}{\sqrt{x^2 + 1}} + c$

Solución b)

Consideremos $x = \tan t$:

$$\begin{aligned}x &= \tan t, \\x^2 + 1 &= \sec^2 t, \\dx &= \sec^2 t dt\end{aligned}$$

por lo anterior:

$$\begin{aligned}\int \frac{dx}{(x^2 + 1)^{\frac{3}{2}}} &= \int \frac{\sec^2 t}{\sec^3 t} dt \\&= \int \cos t dt \\&= \sin t + c \\&= \frac{\tan t}{\sec t} + c \\&= \frac{x}{\sqrt{x^2 + 1}} + c\end{aligned}$$

RESPUESTA B): $\frac{x}{\sqrt{x^2 + 1}} + c$

Ejemplo 1.27

$$1686 \int \frac{x^2 dx}{(8x^3 + 27)^{\frac{2}{3}}}$$

Solución

Considerando $t = 8x^3 + 27$:

$$\begin{aligned}t &= 8x^3 + 27, \\dt &= 24x^2 dx, \\\frac{dt}{24} &= x^2 dx,\end{aligned}$$

sustituyendo en $\int \frac{x^2}{(8x^3 + 27)^{\frac{2}{3}}} dx$:

$$\begin{aligned} \int \frac{x^2 dx}{(8x^3 + 27)^{\frac{2}{3}}} &= \int \frac{1}{(8x^3 + 27)^{\frac{2}{3}}} (x^2 dx) \\ &= \int \frac{1}{t^{\frac{2}{3}}} \frac{dt}{24} \\ &= \frac{1}{24} \int t^{-\frac{2}{3}} dt \\ &= \frac{3}{24} t^{\frac{1}{3}} + c \\ &= \frac{1}{8} (8x^3 + 27)^{\frac{1}{3}} + c \end{aligned}$$

RESPUESTA: $\frac{1}{8} (8x^3 + 27)^{\frac{1}{3}} + c$

Ejemplo 1.28

1688 $\int \frac{dx}{\sqrt{x(1-x)}}$

Solución a)

Completando el cuadrado:

$$\begin{aligned} x(1-x) &= x - x^2 \\ &= \frac{1}{4} - \frac{1}{4} + x - x^2 \\ &= \frac{1}{4} - \left(x - \frac{1}{2}\right)^2, \end{aligned}$$

por lo anterior se tiene:

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}},$$

aplicando la regla $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right)$ con $a = \frac{1}{2}$ y $u = x - \frac{1}{2}$:

$$\begin{aligned} \int \frac{dx}{\sqrt{x(1-x)}} &= \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}} \\ &= \arcsin\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right) + c \\ &= \arcsin(2x - 1) + c \end{aligned}$$

RESPUESTA A): $\arcsin(2x-1) + c$

Solución b)

Considerando:

$$\sqrt{x(1-x)} = \sqrt{x}\sqrt{1-x},$$

proponemos el cambio de variable $t^2 = x$:

$$t^2 = x,$$

$$2t dt = dx,$$

$$t = \sqrt{x},$$

sustituyendo en $\int \frac{dx}{\sqrt{x(1-x)}}$:

$$\begin{aligned} \int \frac{dx}{\sqrt{x(1-x)}} &= \frac{dx}{\sqrt{x}\sqrt{1-x}} \\ &= \int \frac{2t dt}{t\sqrt{1-t^2}} \\ &= 2 \int \frac{dt}{\sqrt{1-t^2}} \\ &= 2 \arcsin t + c \\ &= 2 \arcsin(\sqrt{x}) + c \end{aligned}$$

RESPUESTA B): $2 \arcsin(\sqrt{x}) + c$

Ejemplo 1.29

1690 $\int \frac{e^x}{2+e^x} dx$

Solución

Considerando $t = 2 + e^x$:

$$t = 2 + e^x,$$

$$dt = e^x dx,$$

sustituyendo en $\int \frac{e^x}{2+e^x} dx$:

$$\begin{aligned} \int \frac{e^x}{2+e^x} dx &= \int \frac{dt}{t} \\ &= \ln t + c \\ &= \ln(e^x + 2) \end{aligned}$$

RESPUESTA: $\ln(e^x + 2) + c$

Ejemplo 1.30

1692 $\int \frac{dx}{\sqrt{1+e^{2x}}}$

Solución a)

Desarrollando:

$$\begin{aligned}\sqrt{1+e^{2x}} &= \sqrt{e^{2x}(e^{-2x}+1)} \\ &= \sqrt{e^{2x}}\sqrt{e^{-2x}+1} \\ &= e^x\sqrt{e^{-2x}+1},\end{aligned}$$

de lo anterior se obtiene:

$$\begin{aligned}\int \frac{1}{\sqrt{1+e^{2x}}} &= \int \frac{1}{e^x\sqrt{1+e^{-2x}}} \\ &= \int \frac{e^{-x}}{\sqrt{(e^{-x})^2+1}} dx\end{aligned}$$

Consideremos $t = e^{-x}$:

$$\begin{aligned}t &= e^{-x}, \\ dt &= -e^{-x}dx, \\ -dt &= e^{-x}dx,\end{aligned}$$

sustituyendo en $\int \frac{e^{-x}dx}{\sqrt{(e^{-x})^2+1}}$:

$$\begin{aligned}\int \frac{e^{-x}dx}{\sqrt{(e^{-x})^2+1}} &= -\int \frac{dt}{\sqrt{t^2+1}} \\ &= -\ln(t + \sqrt{t^2+1}) + c \\ &= -\ln(e^{-x} + \sqrt{e^{-2x}+1}) + c\end{aligned}$$

RESPUESTA A): $-\ln(e^{-x} + \sqrt{e^{-2x}+1}) + c$

Solución b); Cambio de variable propuesto por el alumno Mariano Prado (marzo de 2021)

Consideremos $u^2 = 1 + e^{2x}$:

$$\begin{aligned} u^2 &= 1 + e^{2x}, \\ u &= \sqrt{1 + e^{2x}}, \\ du &= \frac{e^{2x}}{\sqrt{e^{2x} + 1}} dx, \\ \frac{\sqrt{e^{2x} + 1}}{e^{2x}} du &= dx, \end{aligned}$$

por lo anterior:

$$\begin{aligned} \int \frac{dx}{\sqrt{1 + e^{2x}}} &= \int \frac{du}{u^2 - 1} \\ &= \frac{1}{2} \ln \left(\frac{1 - u}{u + 1} \right) + c \\ &= \frac{1}{2} \ln \left(\frac{1 - \sqrt{1 + e^{2x}}}{\sqrt{1 + e^{2x}} + 1} \right) + c \end{aligned}$$

RESPUESTA B): $\frac{1}{2} \ln \left(\frac{1 - \sqrt{1 + e^{2x}}}{\sqrt{1 + e^{2x}} + 1} \right) + c$

Ejemplo 1.31

1694 $\int \frac{dx}{x \ln(x) \ln(\ln x)}$

Solución

Consideremos $t = \ln(\ln(x))$:

$$\begin{aligned} t &= \ln(\ln(x)), \\ dt &= \frac{1}{x \ln x} dx, \end{aligned}$$

sustituyendo en $\int \frac{dx}{x \ln(x) \ln(\ln x)}$:

$$\begin{aligned} \int \frac{dx}{x \ln(x) \ln(\ln x)} &= \int \frac{1}{\ln(\ln x)} \left(\frac{dx}{x \ln x} \right) \\ &= \int \frac{dt}{t} \\ &= \ln t + c \\ &= \ln(\ln(\ln(x))) + c \end{aligned}$$

RESPUESTA: $\ln(\ln(\ln(x))) + c$

Ejemplo 1.32

$$1696 \int \frac{\sin x}{\sqrt{\cos^3 x}} dx$$

Solución

Consideremos $t = \cos x$:

$$t = \cos x,$$

$$dt = -\sin x dx,$$

$$-dt = \sin x dx$$

sustituyendo en $\int \frac{\sin x}{\sqrt{\cos^3 x}} dx$:

$$\begin{aligned} \int \frac{\sin x}{\sqrt{\cos^3 x}} dx &= - \int \frac{dt}{t^{\frac{3}{2}}} \\ &= - \int t^{-\frac{3}{2}} dt \\ &= - \left(-2t^{-\frac{1}{2}} \right) + c \\ &= \frac{2}{\sqrt{t}} + c \\ &= \frac{2}{\sqrt{\cos x}} + c \end{aligned}$$

RESPUESTA: $\frac{2}{\sqrt{\cos x}} + c$

Ejemplo 1.33

$$1698 \int \cot x dx$$

Solución

Reescribiendo:

$$\cot x = \frac{\cos x}{\sin x},$$

por lo cual:

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx,$$

consideremos $t = \sin x$:

$$\begin{aligned} t &= \sin x, \\ dt &= \cos x dx, \end{aligned}$$

sustituyendo en $\int \cot x dx$:

$$\begin{aligned} \int \cot x dx &= \int \frac{\cos x}{\sin x} dx \\ &= \int \frac{dt}{t} \\ &= \ln(\sin x) + c \end{aligned}$$

RESPUESTA: $\ln(\sin x) + c$

Ejemplo 1.34

$$1700 \int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx; \quad a^2 \neq b^2$$

Solución

Consideremos $t = a^2 \sin^2 x + b^2 \cos^2 x$:

$$\begin{aligned} t &= a^2 \sin^2 x + b^2 \cos^2 x, \\ dt &= 2a^2 \sin x \cos x dx - 2b^2 \sin x \cos x dx \\ &= 2(a^2 - b^2) \sin x \cos x dx, \end{aligned}$$

$$\frac{dt}{2(a^2 - b^2)} = \sin x \cos x dx,$$

sustituyendo en $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$:

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx &= \int \frac{1}{\sqrt{t}} \left(\frac{dt}{2(a^2 - b^2)} \right) \\ &= \frac{1}{2(a^2 - b^2)} \int \frac{dt}{\sqrt{t}} \\ &= \frac{1}{2(a^2 - b^2)} (2\sqrt{t}) + c \\ &= \frac{1}{a^2 - b^2} \sqrt{a^2 \sin^2 x + b^2 \cos^2 x} + c \end{aligned}$$

RESPUESTA: $\frac{1}{a^2 - b^2} \sqrt{a^2 \sin^2 x + b^2 \cos^2 x} + c$

Ejemplo 1.35

1700.1 $\int \frac{\sin x}{\sqrt{\cos(2x)}} dx$

Solución

Desarrollando:

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - (1 - \cos^2 x) \\ &= 2\cos^2 x - 1, \end{aligned}$$

por lo cual:

$$\int \frac{\sin x}{\sqrt{\cos(2x)}} dx = \int \frac{\sin x}{\sqrt{2\cos^2 x - 1}} dx,$$

considerando $t = \sqrt{2}\cos x$:

$$\begin{aligned} t &= \sqrt{2}\cos x, \\ dt &= -\sqrt{2}\sin x dx, \\ -\frac{dt}{\sqrt{2}} &= \sin x dx, \end{aligned}$$

sustituyendo en $\int \frac{\sin x}{\sqrt{2\cos^2 x - 1}} dx$:

$$\begin{aligned} \int \frac{\sin x}{\sqrt{2\cos^2 x - 1}} dx &= \int \frac{1}{\sqrt{t^2 - 1}} \left(-\frac{dt}{\sqrt{2}} \right) \\ &= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2 - 1}} \\ &= -\frac{1}{\sqrt{2}} \ln \left(t + \sqrt{t^2 - 1} \right) + c \\ &= -\frac{1}{\sqrt{2}} \ln \left(\sqrt{2}\cos x + \sqrt{2\cos^2 x - 1} \right) + c \\ &= -\frac{1}{\sqrt{2}} \ln \left(\sqrt{2}\cos x + \sqrt{\cos(2x)} \right) + c \end{aligned}$$

RESPUESTA: $-\frac{1}{\sqrt{2}} \ln \left(\sqrt{2} \cos x + \sqrt{\cos(2x)} \right) + c$

Ejemplo 1.36

1700.2 $\int \frac{\cos x}{\sqrt{\cos(2x)}} dx$

Solución

Desarrollando:

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 1 - \sin^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x, \end{aligned}$$

por lo cual:

$$\int \frac{\cos x}{\sqrt{\cos(2x)}} dx = \int \frac{\cos x}{\sqrt{1 - 2\sin^2 x}} dx,$$

considerando $t = \sqrt{2} \sin x$:

$$\begin{aligned} t &= \sqrt{2} \sin x, \\ dt &= \sqrt{2} \cos x dx, \\ \frac{dt}{\sqrt{2}} &= \cos x dx, \end{aligned}$$

sustituyendo en $\int \frac{\cos x}{\sqrt{1 - 2\sin^2 x}} dx$:

$$\begin{aligned} \int \frac{\cos x}{\sqrt{1 - 2\sin^2 x}} dx &= \int \frac{1}{\sqrt{1 - t^2}} \left(\frac{dt}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{1 - t^2}} \\ &= \frac{1}{\sqrt{2}} \arcsin t + c \\ &= \frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \sin x) + c \end{aligned}$$

RESPUESTA: $\frac{1}{\sqrt{2}} \arcsin(\sqrt{2} \sin x) + c$

Ejemplo 1.37

$$1702 \int \frac{dx}{\sin^2 x + 2 \cos^2 x}$$

Solución

Reescribiendo:

$$\sin^2 x + 2 \cos^2 x = \cos^2 x (\tan^2 x + 2),$$

sustituyendo en $\int \frac{dx}{\sin^2 x + 2 \cos^2 x}$:

$$\int \frac{dx}{\sin^2 x + 2 \cos^2 x} = \int \frac{1}{\tan^2 x + 2} \left(\frac{dx}{\cos^2 x} \right),$$

considerando $t = \tan x$:

$$\begin{aligned} t &= \tan x, \\ dt &= \sec^2 x dx, \\ &= \frac{dx}{\cos^2 x}, \end{aligned}$$

por lo anterior:

$$\begin{aligned} \int \frac{1}{\tan^2 x + 2} \left(\frac{dx}{\cos^2 x} \right) &= \int \frac{dt}{t^2 + 2} \\ &= \frac{1}{\sqrt{2}} \arctan \left(\frac{t}{\sqrt{2}} \right) + c \\ &= \frac{1}{\sqrt{2}} \arctan \left(\frac{\tan x}{\sqrt{2}} \right) + c \end{aligned}$$

RESPUESTA: $\frac{1}{\sqrt{2}} \arctan \left(\frac{\tan x}{\sqrt{2}} \right) + c$

Ejemplo 1.38

$$1704 \int \frac{dx}{\cos x}$$

Solución

Reescribiendo:

$$\begin{aligned}\frac{1}{\cos x} &= \sec x \\ &= \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \\ &= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x},\end{aligned}$$

de lo anterior:

$$\int \frac{dx}{\cos x} = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx,$$

considerando $t = \sec x + \tan x$:

$$\begin{aligned}t &= \sec x + \tan x \\ dt &= (\sec^2 x + \sec x \tan x) dx,\end{aligned}$$

sustituyendo en $\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$:

$$\begin{aligned}\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx &= \int \frac{dt}{t} \\ &= \ln t + c \\ &= \ln(\sec x + \tan x) + c\end{aligned}$$

RESPUESTA: $\ln(\sec x + \tan x) + c$

Ejemplo 1.39

$$1710 \int \frac{dx}{(\arcsin x)^2 \sqrt{1-x^2}}$$

Solución

Consideremos $t = \arcsin x$:

$$\begin{aligned}t &= \arcsin x, \\ dt &= \frac{dx}{\sqrt{1-x^2}},\end{aligned}$$

sustituyendo en $\int \frac{dx}{(\arcsin x)^2 \sqrt{1-x^2}}$:

$$\begin{aligned} \int \frac{dx}{(\arcsin x)^2 \sqrt{1-x^2}} &= \int \frac{dt}{t^2} \\ &= -\frac{1}{t} + c \\ &= -\frac{1}{\arcsin x} + c \end{aligned}$$

RESPUESTA: $-\frac{1}{\arcsin x} + c$

Ejemplo 1.40

1712 $\int \frac{x^2+1}{x^4+1} dx$

Solución

Desarrollando:

$$\begin{aligned} \frac{x^2+1}{x^4+1} &= \frac{x^2 \left(1 + \frac{1}{x^2}\right)}{x^2 \left(x^2 + \frac{1}{x^2}\right)} \\ &= \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} - 2 + 2} \\ &= \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2}, \end{aligned}$$

de lo anterior se tiene:

$$\int \frac{x^2+1}{x^4+1} dx = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx,$$

considerando $t = x - \frac{1}{x}$:

$$\begin{aligned} t &= x - \frac{1}{x}, \\ dt &= \left(1 + \frac{1}{x^2}\right) dx, \end{aligned}$$

sustituyendo en $\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx$:

$$\begin{aligned} \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 2} &= \int \frac{dt}{t^2 + 2} \\ &= \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} + c \\ &= \frac{1}{\sqrt{2}} \arctan \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) + c \\ &= \frac{1}{\sqrt{2}} \arctan \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c \end{aligned}$$

RESPUESTA: $\frac{1}{\sqrt{2}} \arctan \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c$

Ejemplo 1.41

1714 $\int \frac{x^4}{(x^5 + 1)^4} dx$

Solución

Consideremos $t = x^5 + 1$:

$$\begin{aligned} t &= x^5 + 1, \\ dt &= 5x^4 dx, \\ \frac{dt}{5} &= x^4 dx, \end{aligned}$$

sustituyendo en $\int \frac{x^4}{(x^5 + 1)^4} dx$:

$$\begin{aligned} \int \frac{x^4}{(x^5 + 1)^4} dx &= \int \frac{1}{t^4} \left(\frac{dt}{5} \right) \\ &= \frac{1}{5} \int \frac{dt}{t^4} \\ &= \frac{1}{5} \left(-\frac{1}{3t^3} \right) + c \\ &= -\frac{1}{15t^3} + c \\ &= -\frac{1}{15(x^5 + 1)^3} + c \end{aligned}$$

RESPUESTA: $-\frac{1}{15(x^5+1)^3} + c$

Ejemplo 1.42

1716 $\int \frac{1}{1-x^2} \ln\left(\frac{1+x}{1-x}\right) dx$

Solución

Consideremos $t = \frac{1+x}{1-x}$:

$$\begin{aligned} t &= \frac{1+x}{1-x}, \\ dt &= \frac{2}{(1-x)^2} dx, \\ \frac{dt}{2} &= \frac{1}{(1-x)^2} dx, \end{aligned}$$

sustituyendo en $\int \frac{1}{1-x^2} \ln\left(\frac{1+x}{1-x}\right) dx$:

$$\begin{aligned} \int \frac{1}{1-x^2} \ln\left(\frac{1+x}{1-x}\right) dx &= \int \ln t \left(\frac{dt}{2}\right) \\ &= \frac{1}{2} \int \ln t dt \\ &= \frac{1}{2} (t \ln t - t) + c \\ &= \frac{1}{2} \left(\frac{1+x}{1-x} \ln\left(\frac{1+x}{1-x}\right) - \frac{1+x}{1-x} \right) + c \end{aligned}$$

RESPUESTA: $\frac{1}{2} \left(\frac{1+x}{1-x} \ln\left(\frac{1+x}{1-x}\right) - \frac{1+x}{1-x} \right) + c$

Ejemplo 1.43

1718 $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Solución

Desarrollando:

$$\begin{aligned}\frac{\sin x \cos x}{\sin^4 x + \cos^4 x} &= \frac{\sin x \cos x}{\cos^4 x (\tan^4 x + 1)} \\ &= \frac{\sin x}{\cos^3 x (\tan^4 x + 1)} \\ &= \frac{\sin x}{\cos x} \frac{1}{\cos^2 x} \left(\frac{1}{\tan^4 x + 1} \right) \\ &= \frac{\tan x}{\cos^2 x} \left(\frac{1}{1 + (\tan^2 x)^2} \right),\end{aligned}$$

por lo cual:

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{1}{1 + (\tan^2 x)^2} \left(\frac{\tan x}{\cos^2 x} dx \right),$$

consideremos $t = \tan^2 x dx$:

$$\begin{aligned}t &= \tan^2 x dx, \\ dt &= \frac{2 \tan x}{\cos^2 x}, \\ \frac{dt}{2} &= \frac{\tan x}{\cos^2 x},\end{aligned}$$

sustituyendo en $\int \frac{1}{1 + (\tan^2 x)^2} \left(\frac{\tan x}{\cos^2 x} dx \right)$:

$$\begin{aligned}\int \frac{1}{1 + (\tan^2 x)^2} \left(\frac{\tan x}{\cos^2 x} dx \right) &= \int \frac{dt}{2(1 + t^2)} \\ &= \frac{1}{2} \int \frac{dt}{1 + t^2} \\ &= \frac{1}{2} \arctan(\tan^2 x) + c\end{aligned}$$

RESPUESTA: $\frac{1}{2} \arctan(\tan^2 x) + c$

Ejemplo 1.44

$$1720 \int \frac{xdx}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}}$$

Solución

Desarrollando:

$$\begin{aligned}
 1+x^2+(1+x^2)^{\frac{3}{2}} &= 1+x^2+\sqrt{(1+x^2)^2(1+x^2)} \\
 &= 1+x^2+\sqrt{(1+x^2)^2}\sqrt{1+x^2} \\
 &= 1+x^2+(1+x^2)(1+x^2)^{\frac{1}{2}} \\
 &= (1+x^2)\left(1+(1+x^2)^{\frac{1}{2}}\right)
 \end{aligned}$$

por lo cual:

$$\begin{aligned}
 \int \frac{xdx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}} &= \int \frac{xdx}{\sqrt{(1+x^2)\left(1+(1+x^2)^{\frac{1}{2}}\right)}} \\
 &= \int \frac{1}{\sqrt{1+(1+x^2)^{\frac{1}{2}}}} \left(\frac{x}{\sqrt{1+x^2}}dx\right),
 \end{aligned}$$

consideremos $t = 1 + (1+x^2)^{\frac{1}{2}}$:

$$\begin{aligned}
 t &= 1 + (1+x^2)^{\frac{1}{2}}, \\
 dt &= \frac{x}{\sqrt{1+x^2}}dx,
 \end{aligned}$$

sustituyendo en $\int \frac{1}{\sqrt{1+(1+x^2)^{\frac{1}{2}}}} \left(\frac{x}{\sqrt{1+x^2}}dx\right)$:

$$\begin{aligned}
 \int \frac{1}{\sqrt{1+(1+x^2)^{\frac{1}{2}}}} \left(\frac{x}{\sqrt{1+x^2}}dx\right) &= \int \frac{dt}{\sqrt{t}} \\
 &= 2\sqrt{t} + c \\
 &= 2\sqrt{1+(1+x^2)^{\frac{1}{2}}} + c
 \end{aligned}$$

RESPUESTA: $2\sqrt{1+(1+x^2)^{\frac{1}{2}}} + c$

Ejemplo 1.45

1722 $\int \frac{1+x}{1-x} dx$

Solución

Desarrollando:

$$\frac{1+x}{1-x} = -1 - \frac{2}{x-1},$$

por lo cual:

$$\begin{aligned} \int \frac{1+x}{1-x} dx &= \int \left(-1 - \frac{2}{x-1} \right) dx \\ &= -\int dx - 2 \int \frac{dx}{x-1} \\ &= -x - 2 \ln(x-1) + c \end{aligned}$$

RESPUESTA: $-x - 2 \ln(x-1) + c$

Ejemplo 1.46

$$1724 \int \frac{x^3}{3+x} dx$$

Solución

Desarrollando:

$$\frac{x^3}{3+x} = x^2 - 3x + 9 - \frac{27}{x+3},$$

por lo cual:

$$\begin{aligned} \int \frac{x^3}{3+x} dx &= \int \left(x^2 - 3x + 9 - \frac{27}{3+x} \right) dx \\ &= \int x^2 dx - 3 \int x dx - 27 \int \frac{dx}{3+x} + 9 \int dx \\ &= \frac{1}{3}x^3 - \frac{3}{2}x^2 - 27 \ln(3+x) + 9x + c \end{aligned}$$

RESPUESTA: $\frac{1}{3}x^3 - \frac{3}{2}x^2 - 27 \ln(3+x) + 9x + c$

Ejemplo 1.47

$$1726 \int \frac{(2-x)^2}{2-x^2} dx$$

Solución

Desarrollando:

$$\begin{aligned} \frac{(2-x)^2}{2-x^2} &= \frac{4-4x+x^2}{2-x^2} \\ &= -\frac{x^2-4x+4}{x^2-2} \\ &= \frac{4x+6}{x^2-2} - 1, \end{aligned}$$

de lo anterior:

$$\begin{aligned} \int \frac{(2-x)^2}{2-x^2} dx &= \int \left(\frac{4x+6}{x^2-2} - 1 \right) dx \\ &= \int \frac{4x}{x^2-2} dx - 6 \int \frac{dx}{x^2-2} - \int dx \\ &= 2 \int \frac{2x}{x^2-2} - 6 \int \frac{dx}{x^2-2} - \int dx \\ &= 2 \ln(x^2-2) - \frac{6}{2\sqrt{2}} \ln \left(\frac{x-\sqrt{2}}{x+\sqrt{2}} \right) - x + c \end{aligned}$$

$$\text{RESPUESTA: } 2 \ln(x^2-2) - \frac{6}{2\sqrt{2}} \ln \left(\frac{x-\sqrt{2}}{x+\sqrt{2}} \right) - x + c$$

Ejemplo 1.48

$$1728 \int \frac{x^5}{x+1} dx$$

Solución

Desarrollando:

$$\begin{aligned} \frac{x^5}{x+1} &= \frac{x^5+1-1}{x+1} \\ &= \frac{(x+1)(x^4-x^3+x^2-x+1)}{(x+1)} - \frac{1}{x+1} \\ &= (x^4-x^3+x^2-x+1) - \frac{1}{x+1}, \end{aligned}$$

de lo anterior:

$$\begin{aligned}
 \int \frac{x^5}{x+1} dx &= \int \left(x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1} \right) dx \\
 &= \int x^4 dx - \int x^3 dx + \int x^2 dx - \int x dx + \int dx - \int \frac{dx}{x+1} \\
 &= \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln(x+1) + c
 \end{aligned}$$

RESPUESTA: $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln(x+1) + c$

Ejemplo 1.49

1730 $\int x\sqrt{2-5x} dx$

Solución

Considerando $t = 2 - 5x$:

$$\begin{aligned}
 t &= 2 - 5x, \\
 -\frac{t-2}{5} &= x, \\
 dt &= -5dx, \\
 -\frac{dt}{5} &= dx,
 \end{aligned}$$

sustituyendo en $\int x\sqrt{2-5x}dx$:

$$\begin{aligned}
 \int x\sqrt{2-5x}dx &= \int \left(-\frac{t-2}{5}\right) \sqrt{t} \left(-\frac{dt}{5}\right) \\
 &= \frac{1}{25} \int (t^{\frac{3}{2}} + 2\sqrt{t}) dt \\
 &= \frac{1}{25} \int t^{\frac{3}{2}} dt + \frac{2}{25} \int \sqrt{t} dt \\
 &= \frac{1}{25} \left(\frac{2}{5}\right) t^{\frac{5}{2}} + \frac{2}{25} \left(\frac{2}{3}\right) + c \\
 &= \frac{2}{125} t^{\frac{5}{2}} + \frac{4}{75} t^{\frac{3}{2}} + c \\
 &= \frac{2}{25} t^{\frac{3}{2}} \left(\frac{1}{5}t + \frac{2}{3}\right) + c \\
 &= \frac{2t^{\frac{3}{2}}}{375} (3t + 10) + c \\
 &= \frac{2(2-5x)^{\frac{3}{2}}}{375} (3(2-5x) + 10) + c \\
 &= -\frac{2}{375} (2-5x)^{\frac{3}{2}} (15x-16) + c
 \end{aligned}$$

RESPUESTA: $-\frac{2}{375} (2-5x)^{\frac{3}{2}} (15x-16) + c$

Ejemplo 1.50

1732 $\int x^3 \sqrt[3]{1+x^2} dx$

Solución

Consideremos $t = 1 + x^2$:

$$\begin{aligned}
 t &= 1 + x^2, \\
 t - 1 &= x^2, \\
 dt &= 2x dx, \\
 \frac{dt}{2} &= x dx,
 \end{aligned}$$

sustituyendo en $\int x^3 \sqrt[3]{1+x^2} dx$:

$$\begin{aligned}
 \int x^3 \sqrt[3]{1+x^2} dx &= \int x^2 \sqrt[3]{1+x^2} (x dx) \\
 &= \int (t-1) \sqrt[3]{t} \left(\frac{dt}{2} \right) \\
 &= \int (t-1) t^{\frac{1}{3}} \left(\frac{dt}{2} \right) \\
 &= \frac{1}{2} \int \left(t^{\frac{4}{3}} - t^{\frac{1}{3}} \right) dt \\
 &= \frac{1}{2} \int t^{\frac{4}{3}} dt - \frac{1}{2} \int t^{\frac{1}{3}} dt \\
 &= \frac{1}{2} \left(\frac{3}{7} \right) t^{\frac{7}{3}} - \frac{1}{2} \left(\frac{3}{4} \right) t^{\frac{4}{3}} + c \\
 &= \frac{3}{14} t^{\frac{7}{3}} - \frac{3}{8} t^{\frac{4}{3}} + c \\
 &= \frac{3}{14} (1+x^2)^{\frac{7}{3}} - \frac{3}{8} (1+x^2)^{\frac{4}{3}} + c
 \end{aligned}$$

RESPUESTA: $\frac{3}{14} (1+x^2)^{\frac{7}{3}} - \frac{3}{8} (1+x^2)^{\frac{4}{3}} + c$

Ejemplo 1.51

1734 $\int \frac{dx}{x^2+x-2}$

Solución

Desarrollando:

$$\begin{aligned}
 x^2+x-2 &= (x+2)(x-1), \\
 (x+2)-(x-1) &= 3, \\
 \frac{(x+2)-(x-1)}{3} &= 1, \\
 \frac{1}{x^2+x-2} &= \frac{1}{3} \frac{(x+2)-(x-1)}{(x+2)(x-1)} \\
 &= \frac{x+2}{3(x+2)(x-1)} - \frac{x-1}{3(x+2)(x-1)} \\
 &= \frac{1}{3(x-1)} - \frac{1}{3(x+2)},
 \end{aligned}$$

por lo cual:

$$\begin{aligned}
 \int \frac{dx}{x^2+x-2} &= \int \left(\frac{1}{3(x-1)} - \frac{1}{3(x+2)} \right) dx \\
 &= \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{dx}{x+2} \\
 &= \frac{1}{3} \ln(x-1) - \frac{1}{3} \ln(x+2) + c \\
 &= \frac{1}{3} \ln \left(\frac{x-1}{x+2} \right) + c
 \end{aligned}$$

RESPUESTA: $\frac{1}{3} \ln \left(\frac{x-1}{x+2} \right) + c$

Ejemplo 1.52

1736 $\int \frac{dx}{(x^2-2)(x^2+3)}$

Solución

Desarrollando:

$$\begin{aligned}
 (x^2+3) - (x^2-2) &= 5, \\
 \frac{(x^2+3) - (x^2-2)}{5} &= 1,
 \end{aligned}$$

de lo anterior:

$$\begin{aligned}
 \frac{1}{(x^2-2)(x^2+3)} &= \frac{1}{5} \left(\frac{(x^2+3) - (x^2-2)}{(x^2-2)(x^2+3)} \right) \\
 &= \frac{1}{5} \frac{x^2+3}{(x^2-2)(x^2+3)} - \frac{1}{5} \frac{x^2-2}{(x^2-2)(x^2+3)} \\
 &= \frac{1}{5(x^2-2)} - \frac{1}{5(x^2+3)},
 \end{aligned}$$

por lo cual:

$$\begin{aligned}
 \int \frac{dx}{(x^2-2)(x^2+3)} &= \int \left(\frac{1}{5(x^2-2)} - \frac{1}{5(x^2+3)} \right) dx \\
 &= \frac{1}{5} \int \frac{dx}{x^2-2} - \frac{1}{5} \int \frac{dx}{x^2+3} \\
 &= \frac{1}{10\sqrt{2}} \ln \left(\frac{x-\sqrt{2}}{x+\sqrt{2}} \right) - \frac{1}{5\sqrt{3}} \arctan \left(\frac{x}{\sqrt{3}} \right) + c
 \end{aligned}$$

RESPUESTA: $\frac{1}{10\sqrt{2}} \ln \left(\frac{x-\sqrt{2}}{x+\sqrt{2}} \right) - \frac{1}{5\sqrt{3}} \arctan \left(\frac{x}{\sqrt{3}} \right) + c$

Ejemplo 1.53

1738 $\int \frac{x}{x^4 + 3x^2 + 2} dx$

Solución

Consideremos $t = x^2$:

$$\begin{aligned} t &= x^2, \\ dt &= 2x dx, \\ \frac{dt}{2} &= x dx, \end{aligned}$$

sustituyendo en $\int \frac{x dx}{x^4 + 3x^2 + 2}$:

$$\begin{aligned} \int \frac{x dx}{x^4 + 3x^2 + 2} &= \int \frac{1}{t^2 + 3t + 2} \left(\frac{dt}{2} \right) \\ &= \frac{1}{2} \int \frac{dt}{(t+1)(t+2)}, \end{aligned}$$

ahora consideremos:

$$\begin{aligned} (t+1) - (t+2) &= -1, \\ (t+2) - (t+1) &= 1, \end{aligned}$$

de lo anterior:

$$\begin{aligned} \frac{1}{(t+1)(t+2)} &= \frac{(t+2) - (t+1)}{(t+1)(t+2)} \\ &= \frac{t+2}{(t+1)(t+2)} - \frac{t+1}{(t+1)(t+2)} \\ &= \frac{1}{t+1} - \frac{1}{t+2}, \end{aligned}$$

por lo tanto:

$$\begin{aligned} \frac{1}{2} \int \frac{dt}{(t+1)(t+2)} &= \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dx \\ &= \frac{1}{2} \int \frac{dt}{t+1} - \frac{1}{2} \int \frac{dt}{t+2} \\ &= \frac{1}{2} \ln(t+1) - \frac{1}{2} \ln(t+2) + c \\ &= \frac{1}{2} \ln \left(\frac{x^2+1}{x^2+2} \right) + c \end{aligned}$$