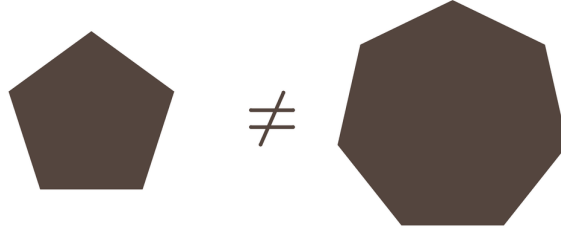


Equal Areas of Regular Polygons

An elementary polygon question, settled by cyclotomic fields

Kevin Martin Cossio Lozano



Theorem. *Two regular convex polygons with positive integer side lengths, measured in the same unit, have equal areas if and only if they have the same number of sides and the same side length.*

Proof. The area of a regular n -gon with side length x is

$$A_n(x) = \frac{nx^2}{4} \cot \frac{\pi}{n}$$

Suppose $A_n(x) = A_m(y)$ for positive integers x, y . If $n = m$, the formula gives $x = y$, so assume $n \neq m$. For $k \geq 3$, put

$$\zeta_k = e^{2\pi i/k}, \quad u_k = \frac{\zeta_k + 1}{\zeta_k - 1} = -i \cot \frac{\pi}{k}$$

Equality of areas gives

$$\frac{u_n}{u_m} = \frac{my^2}{nx^2} \in \mathbb{Q}$$

Since $\zeta_k = (u_k + 1)/(u_k - 1)$, we have $\mathbb{Q}(u_k) = \mathbb{Q}(\zeta_k)$, and consequently

$$\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_m)$$

By Theorem 4.1 of Keith Conrad's *Cyclotomic extensions* [2], distinct indices $n, m \geq 3$ can yield the same cyclotomic field only when one index is twice the other and the smaller is odd. Exchanging the polygons if necessary, $m = 2n$ with n odd. It follows that

$$\frac{\cot(\pi/(2n))}{\cot(\pi/n)} = 1 + \frac{1}{\cos(\pi/n)} \in \mathbb{Q}$$

Thus $2 \cos(\pi/n) = \zeta_{2n} + \zeta_{2n}^{-1}$ is a rational algebraic integer, hence an integer. However,

$$1 \leq 2 \cos \frac{\pi}{n} < 2$$

so it must equal 1. Therefore $n = 3$ and $m = 6$. Equal areas for the triangle and hexagon would then require

$$\frac{\sqrt{3}}{4}x^2 = \frac{3\sqrt{3}}{2}y^2 \implies x^2 = 6y^2$$

This is impossible: the exponent of 2 is even on the left and odd on the right. Thus $n = m$ and $x = y$; the converse is immediate. $\square$

References

- [1] K. M. Cossio Lozano, *Equal volumes of regular polytopes with commensurable edges*. Read article.
- [2] K. Conrad, *Cyclotomic extensions*, expository notes, Theorem 4.1. Read notes.
- [3] A. Berger, “On linear independence of trigonometric numbers,” *Carpathian J. Math.* **34** (2018), no. 2, 157–166. doi:10.37193/CJM.2018.02.04.